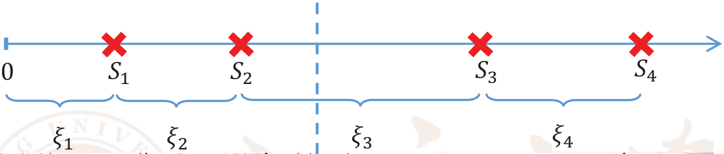
第二章、跳过程

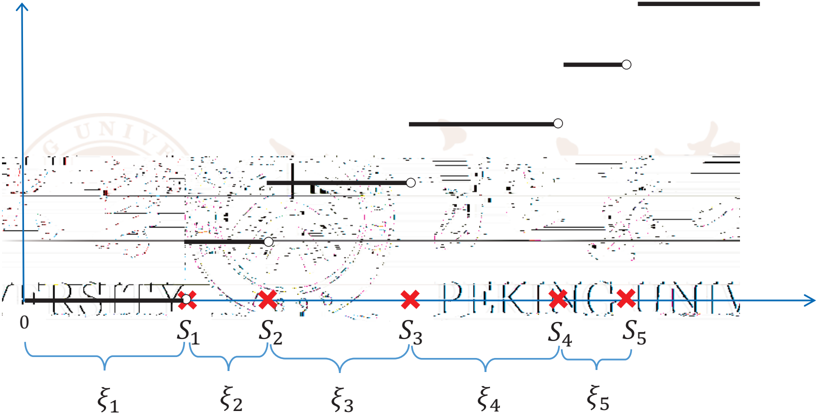
§2.1 泊松过程(Poisson Process)



$X_t = 2$

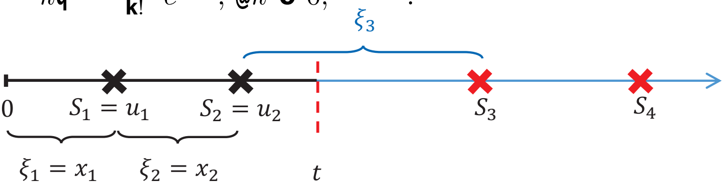
- $\xi_1, \xi_2, \dots$ i.i.d. $\sim$ Exp(λ). $P(\xi > t) = e^{-\lambda t}$.
- $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$, 其中$S_0 := 0$.
-

轨道图: $X_t := |\Xi \cap [0,t]|$, $\Xi = \{S_1, S_2, \ldots\}$.



- $\forall t \geq 0$, $P(t \in \Xi) = P(\bigcup_n \{S_n = t\}) \leq \sum_{n=1}^{\infty} P(S_n = t) = 0$.

验证 $P(|Z|=k)=\frac{(\lambda t)^k}{k!}e^{-\lambda t}, \forall k\geq 0$; “……”.



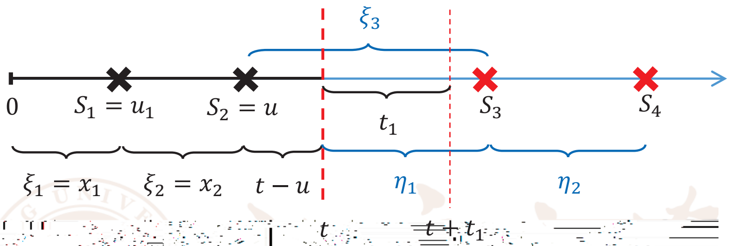
- 假设 $k\geq 1$, $0<u_1<\dots<u_k<t$,

  $P(S_1\in u_1,\dots,S_k\in u_k)=p_{S_1,\dots,S_k}(u_1,\dots,u_k)du_1\cdots du_k$.
- $P(|Z|=k;A)=P(A,\xi_{k+1}>t-S_k)=e^{-\lambda t}e^{\lambda u_k}P(A)$,

  $p_{S_1,\dots,S_k}(u_1,\dots,u_k)=\lambda^k e^{-\lambda(x_1+\dots+x_k)}=\lambda^k e^{-\lambda u_k}$.
- $P(|Z|=k;A)=\lambda^k e^{-\lambda t}1_{\{0<u_1<\dots<u_k<t\}}du_1\cdots du_k$, $(\Rightarrow \frac{t^k}{k!})$.
- $|Z|\sim\mathcal{P}(\lambda t)$, $P(A\,\big|\,|Z|=k)=\frac{k!}{t^k}1_{\{0<u_1<\dots<u_k<t\}}du_1\cdots du_k$.
- $Z\overset{d}{=}\{U_1,\dots,U_W\}$, 其中 $U_1,U_2,\dots$ i.i.d., $\sim U(0,t)$,

  $W\sim\mathcal{P}(\lambda t)$, 且所有随机变量相互独立.

验证Y „ PPpλq, 且Y, Z 相互独立.



- 令B “ tη_1 ą t_1, η_2 ą $t_2, \ldots, \eta_m$ ą t_mu. 往验证

  Pp|Z| “ k, A; Bq “ Pp|Z| “ k, Aq ˆ $e^{-\lambda(t_1+\ldots+t_m)}$.

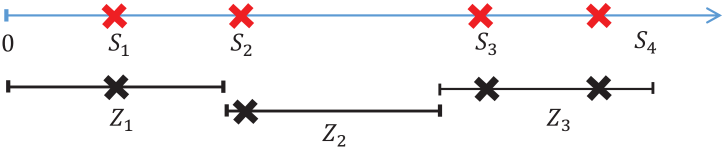
- 令$\hat{B}$ “ tS_{k+1} ą t ` t_1; ξ_{k+2} ą $t_2, \ldots, \xi_{k+m}$ ą t_mu. 则

  t|Z| “ k; Bu “ t|Z| “ k; $\hat{B}$u.

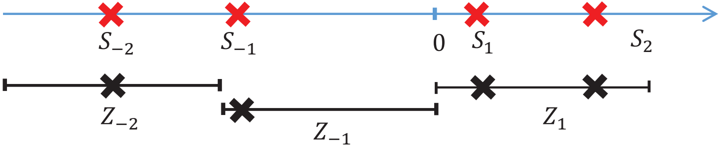
- 左边 “ PpA, S_{k+1} ą t; S_{k+1} ą t ` t_1, $\tilde{B}$q

  “ Pp|X_{t+t_1}| “ k, A; $\tilde{B}$q “ PpA; |Z| “ kq ˆ $e^{-\lambda t_1}$ ¨ Pp$\tilde{B}$q.

构造泊松流(命题2.1.10):

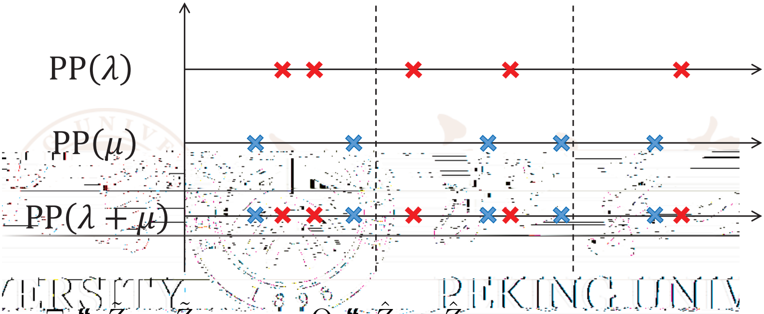


- Z_1 “ Ξ **X** r0, 1s, θ “ θ_1 : Ξ Ñ Y.
- Z_2 “ θpΞq **X** r0, 1s, Z_3 “ $\theta^{(2)}$pΞq **X** r0, 1s, . . .
- $Z_1, Z_2, \ldots$ i.i.d., Ξ “ Z_1 _ Z_2 _ . . . .
- $Z_{\mathbf{n}}$, n **P** $\mathbb{Z}$, i.i.d., Ξ “ . . . _ Z_{-2} _ Z_{-1} _ Z_1 _ Z_2 _ . . . .



泊松流的合并与细分:

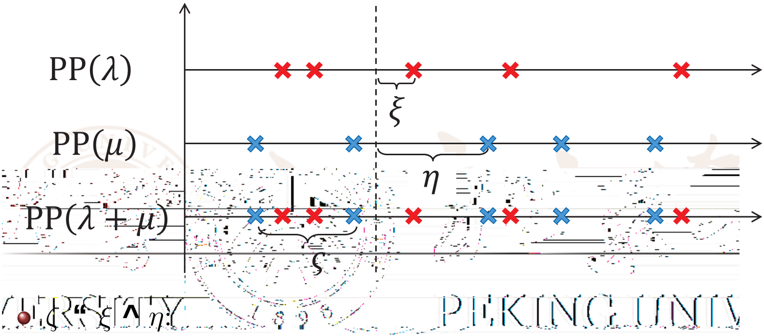
$\Xi \sim PP(\lambda)$ 与 $\Theta \sim PP(\mu)$ 相互独立 vs $\Xi \cup \Theta \sim PF(\lambda+\mu)$.



- $\Xi = \{\tilde{Z}_1 < \tilde{Z}_2 < \dots\}$, $\Theta = \{\hat{Z}_1 < \hat{Z}_2 < \dots\}$,
  $Z_n \overset{d}{=} \{\tilde{U}_1, \dots, \tilde{U}_{\tilde{W}}\}$, $\hat{Z}_n \overset{d}{=} \{\hat{U}_1, \dots, \hat{U}_{\hat{W}}\}$,
- $Z_n = \tilde{Z}_n \cup \hat{Z}_n$, $\tilde{Z}_1 \overset{d}{=} \{U_1, \dots, U_W\}$, $W = \tilde{W} + \hat{W}$.
- $\Xi \cup \Theta = Z_1 < Z_2 < \dots \sim PP(\lambda + \mu)$.

泊松流的合并与细分:

$\Xi \sim PP(\lambda)$ 与 $\Theta \sim PP(\mu)$ 相互独立 vs $\Xi \cup \Theta \sim PF(\lambda + \mu)$.



- $\zeta = \xi \wedge \eta$:

$$P(\zeta > t) = e^{-\lambda t} \cdot e^{-\mu t}, \quad P(\xi < \eta) = \frac{\lambda}{\lambda + \mu} = p.$$

- $\zeta \sim \mathrm{Exp}(\lambda + \mu)$, $V \sim \mathcal{G}(p)$.

$$\zeta_1 + \cdots + \zeta_V \sim \mathrm{Exp}(\lambda).$$

泊松 过程Ξ:

- $\mathbb{R}^d$ 上: (i) $|\Xi\cap D| \sim \mathcal{P}(\lambda|D|)$,
  (ii) 若$D_1,\dots,D_n$ 互不交, 则$|\Xi\cap D_i|$, $i=1,\dots,n$ 独立.
- $(\mathbb{S},\mu(\cdot))$ 上: (i) $|\Xi\cap D|\sim\mathcal{P}(\mu(D))$, (ii).
- 构造:
  (a) 将$\mathbb{S}$划分为$D_1,D_2,\cdots$ 使得$0<\mu(D_n)<\infty$;
  (b) 取$U_{n1},U_{n2},\dots\overset{\text{i.i.d.}}{\sim}\frac{1}{\mu(D_n)}\mu(\cdot)$, $W_n\sim\mathcal{P}(\mu(D_n))$;
  (c) $\Xi=\bigcup_n\{U_{n1},\dots,U_{nW_n}\}$.
- 例: $\mathbb{R}_1$ 上, $\mu((a,b))=\int_a^b f(x)dx$. 则

$$P(\Xi\cap[a,b]=\emptyset)=e^{-\int_a^b f(x)dx}.$$

§2.2 跳过程的构造及其转移概率

1. 定义

- $\forall i \in S$,

  一组小闹钟q_{ij}, $j\neq i$; 或

  大闹钟q_i 与色子$\hat{p}_{ij}$, $j\neq i$:

$$q_i = \sum_{j\neq i} q_{ij} < \infty, \quad \hat{p}_{ij} = \frac{q_{ij}}{q_i}.$$

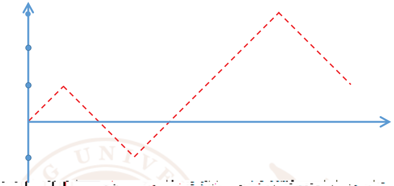
- 特殊情况: 吸收态: $q_i = 0$, $\hat{p}_{ii} := 1$.

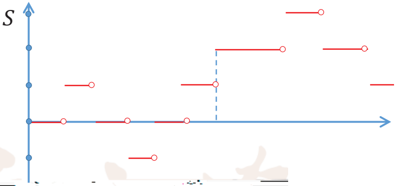
  非吸收态: $q_i > 0$, $\hat{p}_{ii} := 0$.

- 速率矩阵(定义2.2.1):

$$Q = (q_{ij})_{S\times S}, \quad q_{ii} := -q_i.$$

- 色子 $\hat{P} = (\hat{p}_{ij})_{S\times S}$：嵌入链 $\{\hat{X}_n\}$；　　闹钟：时间变换.



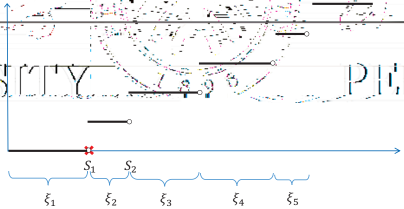


2. 爆炸与非爆炸.

- 命题2.2.3. $\zeta_1, \zeta_2, \cdots$ 独立, $\zeta_n \sim \mathrm{Exp}(\lambda_n)$, $\tau = \sum_n \zeta_n$. 则

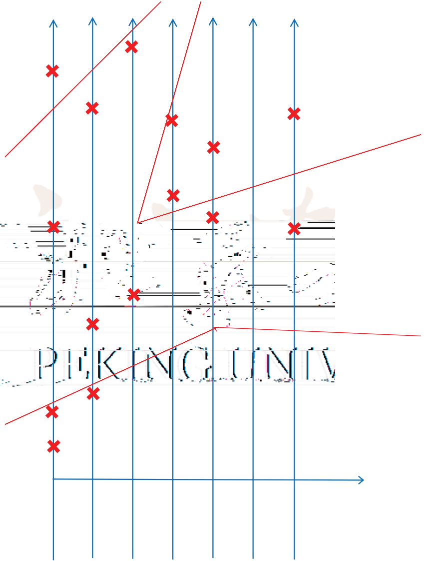
$$\sum_i \frac{1}{\lambda_i} = \infty \quad \text{iff} \quad P(\tau = \infty) = 1,$$

$$\sum_i \frac{1}{\lambda_i} < \infty \quad \text{iff} \quad P(\tau < \infty) = 1.$$

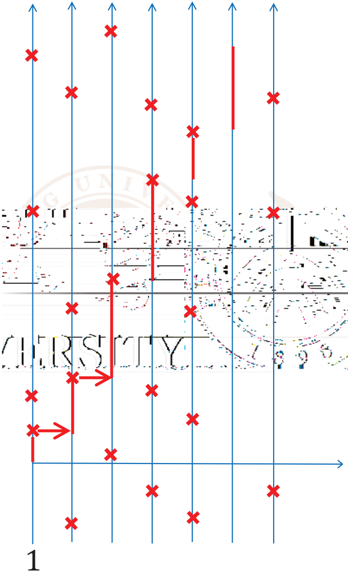


例2.2.4 & 2.2.6. Yule过程/纯生过程. $\lambda_i = i\lambda, i \geq 1$. 非爆炸.

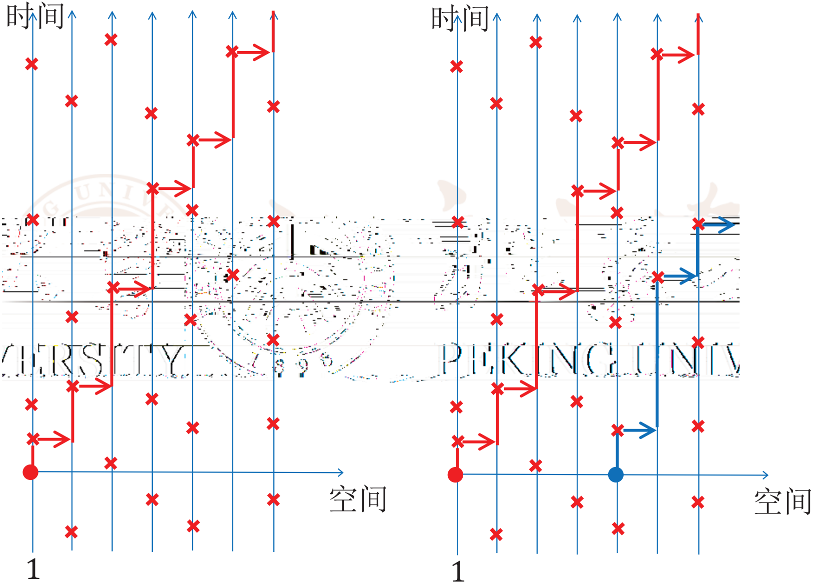
图表示与对偶:



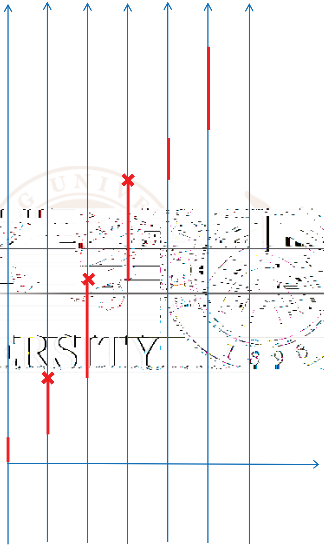
图表示与对偶:



图表示与对偶:

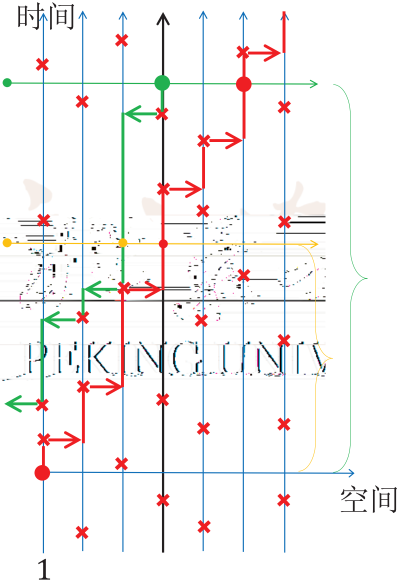


图表示与对偶:



图表示与对偶:

- $X_t \ni i + 1$ iff $Y_t^{(i)} = 0$.
- 概率 $= (1 - e^{-\lambda t})^i$.



3. 转移概率与转移速率.

- 转移概率 $P(t) = (p_{ij}(t))_{S\times S}$:

$$p_{ij}(t) = P_i(X_t = j) = \sum_{n=0}^{\infty} P_i(X_t = j, N_t = n).$$

- $P_i(\hat{X}_1 = i_1, \ldots, \hat{X}_n = i_n; S_n \le t < S_{n+1})$
- $= p_{i_0 i_1} \cdots p_{i_{n-1}, i_n} P(\cdots)$,
- $= \int_{\Delta_n} q_{i_0} e^{-q_{i_0} x_0} \cdots q_{i_n} e^{-q_{i_n} x_n} dx_0 \cdots dx_n$,

$\Delta_n := \{\vec{x} : x_i \ge 0, \forall i; x_0 + \cdots + x_{n-1} \le t < x_0 + \cdots + x_n\}$.

- 非爆炸: $\sum_j p_{ij}(t) = 1, \forall i, t$.

$$\sum_{j\in S} p_{ij}(t) = P_i(X_t \in S) = P_i(\tau_\infty > t) \le 1.$$

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- 后退方程$P'(t) = QP(t)$ 的应用. f 是的函数 .

$$f_t := P(t)f, \quad f_t(i) := \sum_k p_{ik}(t) f(k) = E_i f(X_t),$$

$$f_t' = (P(t)f)' = QP(t)f = Q$$

-

$Q \quad \mu_t Q$

补充知识:

- 定义2.2.19. (强连续)马氏半群$\{\mathbf{P}(t) : t \geq 0\}$:

  (1) $\mathbf{P}(0) = \mathbf{I}$, (2) $\mathbf{P}(t+s) = \mathbf{P}(t)\mathbf{P}(s)$,

  (3) $\lim_{t\to 0} \mathbf{P}(t) = \mathbf{P}(0)$, $p_{ij}(t) \to p_{ij}(0)$, $\forall i, j$.

- 命题2.2.22. $\mathbf{Q} = \mathbf{P}'(0)$ 存在且满足:

  $0 \leq q_{ij} < \infty$; $0 \leq q_i := -q_{ii} \leq \infty$; $\sum_{j\neq i} q_{ij} \leq q_i$.

- 命题2.2.24. $\mathbf{Q}$ 保守，则后退方程$\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$ 成立.

- 连续时间马氏链$\{X_t\}$: 以$\{\mathbf{P}(t), t \geq 0\}$为转移矩阵.

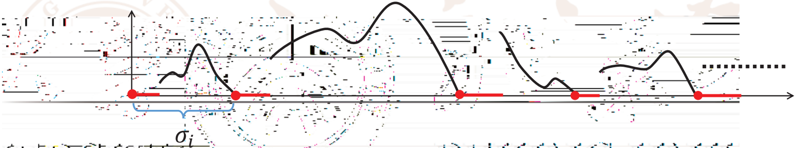
- 生成元: 定义域为$\mathcal{D} = \mathcal{D}(\mathcal{L}) \subset 2^{S}$,

$$\mathcal{L} : \mathcal{D} \to 2^{S}, \quad f \mapsto \mathcal{L}(f) = \mathbf{Q}f,$$

$$\frac{d}{dt} f_t = \mathcal{L}(f_t), \quad f_t(i) := E_i f(X_t).$$

§2.3 首达时、吸收概率

- 不可约：可达、互通. (定义2.3.1, 命题2.3.2)
- 强马氏性(引理2.3.3 & 2.4.2). 令τ “ τ_i 或σ_i. 在τ ă 8 的条件下, tY_t :“ $X_{\tau+t}$u 为从i 出发的跳过程, 且与$X_{[0,\tau]}$ 独立.



- x_i “ P_ipτ_o ă 8q: x_i “ $\sum_j p_{ij} x_j$, i ‰ o; x_o “ 1.
- y_i “ $E_i \tau_o$: y_i “ $\frac{1}{q_i}$ ` $\sum_j p_{ij} y_j$, i ‰ 0; y_o “ 0.
- z_i “ $E_i \int_0^{\tau_D} 1_{\{X_t = o\}} dt$:

  z_i “ $\sum_j p_{ij} z_j$, i R D, i ‰ k; z_o “ $\frac{1}{q_o}$ ` $\sum_i p_{oi} z_i$; $z|_D$ “ 0.

- 常返: $P_i(\forall t, \exists s > t, \text{ s.t. } X_s = i) = 1$.
- 命题2.4.5. i 常返的等价条件:

  (a) $q_i = 0$ 或 $P_i(\sigma_i < \infty) = 1$,

  (b) 嵌入链 $\{\hat{X}_n\}$ 常返,

  (c) 格林函数发散, 即 $G_{ii} = \int p_{ii}(t)dt = \infty$,

  (d) 骨架链 $\{X_{n\delta} : n = 0, 1, 2 \cdots\}$ 常返.

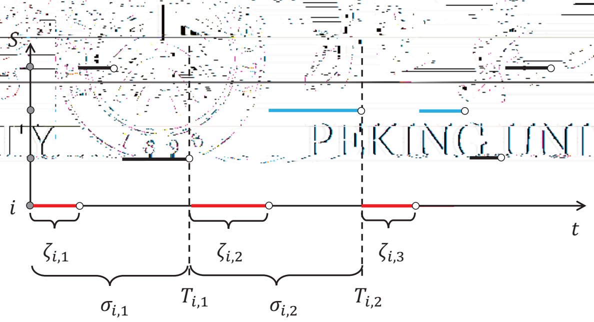
- 推论2.4.3 & 2.4.4. 在 i 的总耗时:

$$\zeta_1 + \cdots + \zeta_{\hat{V}_i}, \qquad \zeta_n \text{ 独立地 } \sim \mathrm{Exp}(q_i).$$

因此, 常返 $\Rightarrow$ 非爆炸, $G_{ij} = \frac{1}{q_j}\hat{G}_{ij}$.

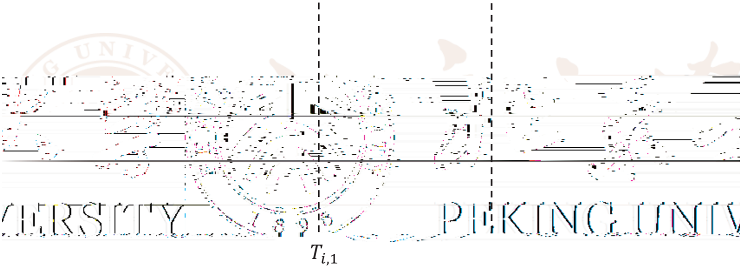
§2.5 不变分布与正常返

- 定义2.5.1. 不变分布/不变测度: $\pi = \pi \mathbf{P}(t)$, $\forall t$.
- 引理2.5.3. 若不变分布存在, 则非爆炸.
- 强马氏性(引理2.5.5): 假设i 常返, $q_i > 0$. 假设$X_0 = i$. 那么, 游弋i.i.d., 从而$(\sigma_{i,n}, \zeta_{i,n})$, $n = 1, 2, \cdots$ i.i.d.



- 命题2.5.8. 假设i 常返且$q_i > 0$. 令$V_i(t) = \int_0^t 1_{\{X_s=i\}} ds$. 则

$$P\left(\lim_{t\to\infty} \frac{1}{t} V_i(t) = \frac{1}{q_i E_i \sigma_i}\right) = 1.$$

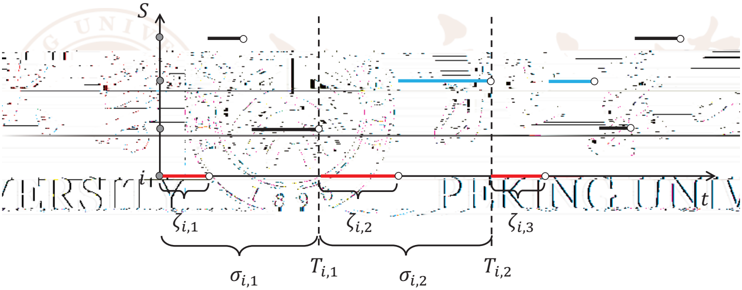


命题 (命题2.5.8)

设不可约、常返. 则$q_{\mathbf{i}}\mu_{\mathbf{i}}$ “ $\hat{\mu}_{\mathbf{i}}$; $\mu\mathbf{Q}$ “ 0; $\lambda\mathbf{Q}$ “ 0 *iff* λ “ $c\mu$.

- $\mu_{\mathbf{i}}$ “ $E_{\mathbf{o}}\int_0^{\sigma} 1_{\{\mathbf{X}_t=\mathbf{i}\}}dt$ “ $\frac{1}{\mathbf{q}_i}$ ˆ $E_{\mathbf{o}}V_{\mathbf{i}}^{(\mathbf{Y})}$, @$i$. σ “ $\sigma_{\mathbf{o}}$.

  ‹‹ “ ζ_1 ` . . . ` $\zeta_{\tilde{\mathbf{V}}_i}$, $\zeta_{\mathbf{n}}$ „ Expp$q_{\mathbf{i}}$q.



- $\frac{1}{\mathbf{q}_i}q_{\mathbf{ij}}$ “ $\hat{p}_{\mathbf{ij}}$, $\lambda\mathbf{Q}$ “ 0 iff

$$\mathrm{p}\lambda_{\mathbf{j}}q_{\mathbf{j}}\mathrm{q} \text{ “ } \sum_{\mathbf{i}\neq\mathbf{j}}\lambda_{\mathbf{i}}q_{\mathbf{ij}} \text{ “ } \sum_{\mathbf{i}\neq\mathbf{j}}\mathrm{p}\lambda_{\mathbf{i}}q_{\mathbf{i}}\mathrm{q}\hat{p}_{\mathbf{ij}},\text{, i.e., } q\lambda \text{ “ } c\hat{\mu}.$$

命题 (命题2.5.9, 推论2.5.10)

- 定义2.5.7. 正常返: $q_i = 0$ 或 $E_i\sigma_i < \infty$.

  常返: $q_i > 0$, $P_i(\sigma_i < \infty) = 1$, 且 $E_i\sigma_i = \infty$.

- 命题2.5.11. 设不可约. 则下面三条等价: (i) 所有状态正常返, (2) 存在正常返态, (3) 存在不变分布. 此时, $\pi Q = 0$,

$$\pi_i = \frac{1}{q_i E_i\sigma_i} = \frac{1}{E_0\sigma_0} E_0 \int_0^{\sigma_0} 1_{\{X_t = i\}} dt \approx i \text{ 的频率}.$$

- 例2.5.4. $\pi Q = 0$, 但π 不是不变分布.

- 遍历(定理2.5.17): 不可约、正常返, $\sum_i \pi_i |f(i)| < \infty$, 则

$$P_\mu\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X_s)ds = \sum_{i\in S}\pi_i f(i)\right) = 1.$$

- 强遍历(定理2.5.18): $\lim_{t\to\infty} p_{ij}(t) = \pi_j$, $\forall j$.

§2.6 可逆分布

- 总假设Q不可约. 若π为测度, 满足$\pi\mathbf{Q}=0$, 令

$$\tilde{q}_{ij} := \frac{\pi_j q_{ji}}{\pi_i}.$$

- 则, $\pi_i\tilde{q}_{ij} = \pi_j q_{ji}$, $\forall i,j$.
- $\tilde{\mathbf{Q}} = (\tilde{q}_{ij})_{S\times S}$ 仍为转移矩阵, 且$\tilde{q}_i = q_i$, $\pi\tilde{\mathbf{Q}} = 0$.
-