

§8.2 N-P引理和似然比检验

- 简单假设检验问题: $\Theta = \{\theta_0, \theta_1\}$.

$$H_0 : \theta = \theta_0 \leftrightarrow H_1 : \theta = \theta_1.$$

- 似然函数: $L(\vec{x}, \theta) = \prod_{i=1}^n f(x_i, \theta)$. (以连续型为例)
- 似然比否定域/似然比检验:

$$\mathcal{W}_\lambda = \{\vec{x} : L(\vec{x}, \theta_1) > \lambda L(\vec{x}, \theta_0)\}.$$

- 定理2.1. (Neyman-Pearson引理) 若 λ_0 使得 $P_{\theta_0}(\vec{X} \in \mathcal{W}_{\lambda_0}) = \alpha$,

$$P_{\theta_0}(\vec{X} \in \mathcal{W}_{\lambda_0}) = \alpha,$$

则 \mathcal{W}_{λ_0} 是水平为 α 的UMP 否定域.

- 定理2.2. 上述 \mathcal{W}_{λ_0} 是无偏否定域, 因而也是UMPU 否定域.

- N-P 引理的证明: 假设 \tilde{W} 也是水平为 α 的否定域, 往验证

$$P_{\theta_1}(\vec{X} \in W) \geq P_{\theta_1}(\vec{X} \in \tilde{W}), \quad \text{其中 } W_{\lambda_0} \triangleq W.$$

- 由假设知, W 与 \tilde{W} 都是水平为 α 的否定域, 即

$$P_{\theta_0}(\vec{X} \in W) = P_{\theta_0}(\vec{X} \in \tilde{W}) = \alpha.$$

- 左右两边同时扣除交事件 $\{\vec{X} \in W \cap \tilde{W}\}$ 的概率, 即, 需验证

$$P_{\theta_0}(\vec{X} \in W \setminus \tilde{W}) = P_{\theta_0}(\vec{X} \in \tilde{W} \setminus W)$$

$$\Rightarrow L \triangleq P_{\theta_1}(\vec{X} \in W \setminus \tilde{W}) \geq P_{\theta_1}(\vec{X} \in \tilde{W} \setminus W) \triangleq R.$$

- 由 $W = W_{\lambda_0} = \{\vec{x} : L(\vec{x}, \theta_1) > \lambda_0 L(\vec{x}, \theta_0)\}$ 知 “ \Rightarrow ” 成立:

$$L = \int_{W \setminus \tilde{W}} L(\vec{x}, \theta_1) d\vec{x} \geq \lambda_0 \int_{W \setminus \tilde{W}} L(\vec{x}, \theta_0) d\vec{x},$$

$$R = \int_{\tilde{W} \setminus W} L(\vec{x}, \theta_1) d\vec{x} \leq \lambda_0 \int_{\tilde{W} \setminus W} L(\vec{x}, \theta_0) d\vec{x}.$$

例2.1. $X \sim N(\mu, 1)$. 求假设检验问题 $H_0: \mu = 0 \leftrightarrow H_1: \mu = 2$ 的水平为 $\alpha = 0.05$ 的UMP 否定域.

- 似然函数与似然比:

$$\frac{L(\vec{x}, \theta_1)}{L(\vec{x}, \theta_0)} = \frac{\frac{1}{\sqrt{2\pi}^n} e^{-\frac{1}{2} \sum_{i=1}^n (x_i - 2)^2}}{\frac{1}{\sqrt{2\pi}^n} e^{-\frac{1}{2} \sum_{i=1}^n x_i^2}} = e^{\frac{1}{2} \sum_{i=1}^n 4(x_i - 1)}.$$

- 似然比否定域:

$$\mathcal{W}_\lambda = \left\{ \vec{x} : \frac{L(\vec{x}, \theta_1)}{L(\vec{x}, \theta_0)} > \lambda \right\} = \{ \vec{x} : \bar{x} > c \}.$$

- $T(\vec{x}_1, \dots, \vec{x}_n)$ 且 k 称为检验统计量

- 根据 α 选择 λ (等价地, 选择 c):

$$\alpha = P_{\theta_0}(\bar{X} > c) = P(Z > c\sqrt{n}) \Rightarrow c = z_{1-\alpha}/\sqrt{n}.$$

- 查表获得 $z_{1-0.05} = 1.65$. 从而所求为

$$\mathcal{W} = \{ \vec{x} : \bar{x} > 1.65/\sqrt{n} \}.$$

例2.2. $\Theta = \{0, 1\}$. $f(x, 0) = 1_{\{0 < x < 1\}}$, $f(x, 1) = 2x1_{\{0 < x < 1\}}$. 求假设检验问题 $H_0 : \theta = 0 \leftrightarrow H_1 : \theta = 1$ 的UMP 否定域.

- 似然函数与似然比:

$$\frac{L(\vec{x}, \theta_1)}{L(\vec{x}, \theta_0)} = \frac{2^n x_1 \cdots x_n 1_{\{0 < x_1, \dots, x_n < 1\}}}{1_{\{0 < x_1, \dots, x_n < 1\}}} = 2^n x_1 \cdots x_n 1_{\{0 < x_1, \dots, x_n < 1\}}.$$

- 似然比否定域与检验统计量 $T = T(x_1, \dots, x_n)$:

$$W_\lambda = \left\{ \vec{x} : \frac{L(\vec{x}, \theta_1)}{L(\vec{x}, \theta_0)} > \lambda \right\} = \left\{ \vec{x} : -2 \sum_{i=1}^n \ln x_i < c \right\}.$$

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