§7.6 估计的渐近分布

- 定义6.2. 设$T_n = T_n(X_1, \cdots, X_n)$ 满足:

$$\sqrt{n}(T_n - g(\theta)) \xrightarrow{d} Z \sim N(0, \sigma^2), \quad \forall \theta \in \Theta,$$

则称T_n 是渐近正态的, 其中 $\sigma^2 = \sigma^2_\theta$ 称为渐进方差.

- 工具: CLT & Δ 方法.
- 定理6.3 (Δ 方法). 设 $\sqrt{n}(T_n - \theta) \xrightarrow{d} Z \sim N(0, \sigma^2)$, $h'(\theta)$ 存在且不为0, 则

$$\sqrt{n}\big(h(T_n) - h(\theta)\big) \xrightarrow{d} W \sim N(0, h'(\theta)^2 \sigma^2).$$

例6.1. 总体: $X \sim N(\mu, \sigma^2)$, 样本量: n.

- UMVU 估计: $\hat{\mu} = \bar{X}$, $\widehat{\sigma^2} = S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.
- $\hat{\mu}$ 渐近正态: 事实上,

$$\sqrt{n}(\bar{X} - \mu) \sim N(0, \sigma^2).$$

- 定理7.1. $(n-1)S^2 \overset{d}{=} \sigma^2 K_{n-1}$, 其中, $K_{n-1} \sim \chi^2(n-1)$.
- S^2 渐近正态: CLT,

$$\frac{\sum_{i=1}^{n-1} Z_i^2 - (n-1)}{\sqrt{n-1}} \xrightarrow{d} W \sim N(0, \mathrm{var}(Z^2))$$

$$\sqrt{n}(S^2 - \sigma^2) \approx \sqrt{n-1}(S^2 - \sigma^2) \xrightarrow{d} \sigma^2 W \sim N(0, 2\sigma^4).$$

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例6.3 (续). 总体: $X \sim N(\mu,1)$, 待估量: $g(\mu)=\Phi(x_0-\mu)$.

- $\hat{\mu}=\bar{X}$ 是完全充分统计量, 但$g(\hat{\mu})$ 不是$g(\mu)$ 的无偏估计.
- 令$h(\mu)=\Phi\left(\sqrt{\frac{n}{n-1}}(x_0-\mu)\right)$.
- 记$p_{\mu,\sigma^2}(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, 则

$$
\begin{aligned}
E_\mu h(\hat{\mu}) &= E_\mu \Phi\left(\sqrt{\frac{n}{n-1}}(x_0-\bar{X})\right)\\
&=\int_{-\infty}^{\infty} p_{\mu,\frac{1}{n}}(y)\Phi\left(\sqrt{\frac{n}{n-1}}(x_0-y)\right)dy\\
&=\int_{-\infty}^{\infty} p_{\mu,\frac{1}{n}}(y)\int_{-\infty}^{\sqrt{\frac{n}{n-1}}(x_0-y)} p_{0,1}(z)dzdy\\
&=P\left(Z\le\sqrt{\frac{n}{n-1}}(x_0-Y)\right),
\end{aligned}
$$

其中, Y,Z 相互独立, $Y\sim N(\mu,\frac{1}{n})$, $Z\sim N(0,1)$.

例6.3 (续). 总体: $X \sim N(\mu,1)$, 待估量: $g(\mu)=\Phi(x_0-\mu)$.

- 已有: 取Y, Z 相互独立, $Y \sim N(\mu,\frac{1}{n})$, $Z \sim N(0,1)$, 则

$$E_\mu h(\hat{\mu}) = P\left(Z \le \sqrt{\frac{n}{n-1}}(x_0 - Y)\right),$$

- $\sqrt{\frac{n-1}{n}}Z + Y \sim N(\mu,1)$. 因此,

$$E_\mu h(\hat{\mu}) = P\left(\sqrt{\frac{n-1}{n}}Z + Y - \mu \le x_0 - \mu\right) = \Phi(x_0 - \mu).$$

- $\hat{\mu}$ 是完全充分统计量, $h(\hat{\mu})$ 是$g(\mu)$ 的无偏估计, 因此是UMVU 估计.