§7.3 估计的无偏性

- 定义3.1. 若统计量$\mathbf{T} = \mathbf{T}(\mathbf{X}_1, \cdots, \mathbf{X}_n)$ 满足

$$\mathbf{E}_\theta \mathbf{T} = \mathbf{g}(\boldsymbol{\theta}), \quad \forall \boldsymbol{\theta} \in \Theta.$$

则称$\mathbf{T}$ 为$\mathbf{g}(\boldsymbol{\theta})$ 的无偏估计.

- 例3.1. 样本均值$\overline{\mathbf{X}}$ 是期望$\boldsymbol{\mu}$ 的无偏估计.

$$\begin{aligned}\mathbf{E}_\theta \overline{\mathbf{X}} &= \mathbf{E}_\theta \frac{1}{\mathbf{n}}(\mathbf{X}_1 + \cdots + \mathbf{X}_n) \\ &= \frac{1}{\mathbf{n}}(\mathbf{E}_\theta \mathbf{X}_1 + \cdots + \mathbf{E}_\theta \mathbf{X}_n) = \boldsymbol{\mu}.\end{aligned}$$

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例3.2. 总体: $\mathbf{X} \sim \mathrm{Exp}(\boldsymbol{\lambda})$. 样本量: $\mathbf{n}$. 寻找$\boldsymbol{\lambda}$ 的无偏估计.

- 最大似然估计& 矩估计: $\hat{\boldsymbol{\lambda}} = 1/\bar{\mathbf{X}} = \frac{n}{S_n}$, 其中,

$$\mathbf{S}_n = \mathbf{X}_1 + \cdots + \mathbf{X}_n \sim \Gamma(\mathbf{n}, \boldsymbol{\lambda}), \quad \mathbf{p}_{S_n}(\mathbf{y}) = \frac{\boldsymbol{\lambda}^n}{\Gamma(\mathbf{n})}\mathbf{y}^{n-1}\mathbf{e}^{-\lambda y}, \ \mathbf{y} > 0.$$

- 于是,

$$\mathbf{E}\hat{\boldsymbol{\lambda}} = \mathbf{E}\frac{\mathbf{n}}{\mathbf{S}_n} = \mathbf{n}\int_0^\infty \frac{1}{\mathbf{y}} \cdot \frac{\boldsymbol{\lambda}^n}{\Gamma(\mathbf{n})}\mathbf{y}^{n-1}\mathbf{e}^{-\lambda y}\mathbf{dy}$$

$$= \mathbf{n}\frac{\boldsymbol{\lambda}\Gamma(\mathbf{n}-1)}{\Gamma(\mathbf{n})}\int_0^\infty \frac{\boldsymbol{\lambda}^{n-1}}{\Gamma(\mathbf{n}-1)}\mathbf{y}^{n-2}\mathbf{e}^{-\lambda y}\mathbf{dy} = \frac{\mathbf{n}}{\mathbf{n}-1}\boldsymbol{\lambda}.$$