

- 定义6.3. 若存在 $p(x_1, \dots, x_n)$ 使得对任意 n 维矩形 D 都有

$$P(\xi \in D) = \int \cdots \int_D p(x_1, \dots, x_n) dx_1 \cdots dx_n,$$

则称 ξ 为连续型随机向量, 称 $p(x_1, \dots, x_n)$ 为 ξ 的联合密度, 也记为 $p_{\mathbf{X}_1, \dots, \mathbf{X}_n}$! (注: ** 对一般的 D 都成立.)

定义6.4: 对任意 $1 \leq i \leq n$, $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$

例6.1 (项分布). 设 U_1, \dots, U_n 相互独立, 都服从如下分布:

$$P(U_i = k) = p_k, \quad k = 1, \dots, t,$$

其中 $t \geq 2, p_k \geq 0, \forall k$ 且 $p_1 + \dots + p_t = 1$.

- 背景模型: n 次独立重复试验(投掷一枚 t 面骰子). 记

$$X_k = |\{1 \leq i \leq n : U_i = k\}| = \sum_{i=1}^n 1_{\{U_i = k\}}.$$

- $\xi = (X_1, \dots, X_t)$ 的联合分布列:

$$P(\xi = (i_1, \dots, i_t)) = \frac{n!}{i_1! \cdots i_t!} p_1^{i_1} \cdots p_t^{i_t}.$$

- 因为 $X_t = n - \sum_{s=1}^{t-1} X_s$, 所以 ξ 与 (X_1, \dots, X_{t-1}) 等价.
- 任意边缘都服从 项分布.

例, (X_1, X_2) 服从三项分布; 特别地, X_k 服从二项分布:

若 $U_i = k$, 则令 $V_i = 1$; 若 $U_i \neq k$, 则令 $V_i = 0$.

- 定义6.5. 若对任意 $a_i, b_i, i = 1, \dots, n$ 都有

$$P(a_1 \leq X_1 \leq b_1, \dots, a_n \leq X_n \leq b_n) \\ = P(a_1 \leq X_1 \leq b_1) \cdots P(a_n \leq X_n \leq b_n),$$

则称 X_1, \dots, X_n 相互独立.

- 若 X_1, \dots, X_n 相互独立, 且 $F_{X_i} = F_{X_j}, i = 2, \dots, n$, 则称 X_1, \dots, X_n 独立同分布.

- 若相互独立, 则上式中的 $a_i \leq X_i \leq b_i$ 可以改为 $X_i \in B_i$.

- 相互独立的充要条件与充分条件:

$$F_{\mathbf{X}_1, \dots, \mathbf{X}_n}(x_1, \dots, x_n) = F_{\mathbf{X}_1}(x_1) \cdots F_{\mathbf{X}_n}(x_n) = f_1(x_1) \cdots f_n(x_n).$$

- 离散型:

$$\begin{aligned} & P\left(X_1 = x_{i_1}^{(1)}, \dots, X_n = x_{i_n}^{(n)}\right) \\ &= P\left(X_1 = x_{i_1}^{(1)}\right) \cdots P\left(X_n = x_{i_n}^{(n)}\right) = p_{i_1}^{(1)} \cdots p_{i_n}^{(n)} \end{aligned}$$

- 连续型(定理6.1):

$$p_{\mathbf{X}_1, \dots, \mathbf{X}_n}(x_1, \dots, x_n) = p_{\mathbf{X}_1}(x_1) \cdots p_{\mathbf{X}_n}(x_n) = p_1(x_1) \cdots p_n(x_n).$$

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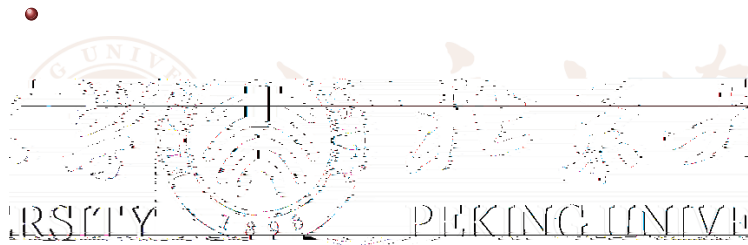
2. n 维随机向量 $\xi = (X_1, \dots, X_n)$ 的数字特征

- 定义6.6. 称 (EX_1, \dots, EX_n) 为 ξ 的期望, 记为 $E\xi$.
- 定义6.7. 记 $\sigma_{ij} = \text{cov}(X_i, X_j)$, $\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}$.
称 $\Sigma = (\sigma_{ij})_{n \times n}$, $\mathbf{R} = (\rho_{ij})_{n \times n}$ 为 ξ 的协方差阵, 相关系数阵.
- 定义6.8. n 维正态分布. 假设 ξ 有如下的联合密度, 则称 ξ 服从 n 维正态分布, 记为 $\xi \sim N(\vec{\mu}, \Sigma)$.

$$p(\vec{x}) = \frac{1}{\sqrt{2\pi}^n \sqrt{|\Sigma|}} \exp \left\{ -\frac{1}{2} (\vec{x} - \vec{\mu}) \Sigma^{-1} (\vec{x} - \vec{\mu})^T \right\}$$

- $n = 1$ 与 $n = 2$ 的特例已介绍.
- $N(\vec{\mu}, \Sigma)$ 的数字特征: $\mu_i = EX_i$, $\sigma_{ij} = \text{cov}(X_i, X_j)$.
- X_1, \dots, X_n 相互独立当且仅当 $\sigma_{ij} = 0, \forall i \neq j$.
- 边缘分布, 条件分布都是正态.

3. n 个随机变量的函数 $Y = f(X_1, \dots, X_n)$



例6.3, 6.4, 定义6.9. 若 X 与 Y 独立, $X \sim \Gamma(r, \lambda)$, $Y \sim \Gamma(s, \lambda)$. 则

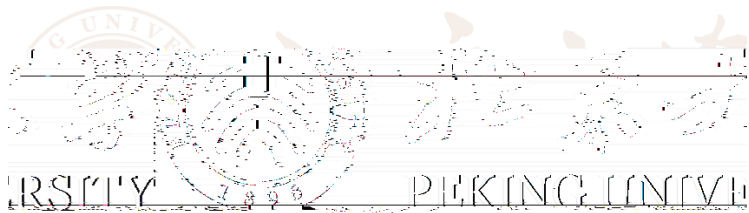
$$X + Y \sim \Gamma(r + s, \lambda).$$

• 密度:

$$p_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, \quad x > 0.$$

• $Z = X + Y$: $p_Z(z) = \int_0^z p_X(x) p_Y(z-x) dx, \quad \forall z > 0,$

$$\begin{aligned} p_Z(z) &= C \int_0^z x^{r-1} e^{-\lambda x} \cdot (z-x)^{s-1} e^{-\lambda(z-x)} dx \\ &= C e^{-\lambda z} \int_0^1 (tz)^{r-1} ((1-t)z)^{s-1} d(tz) = \hat{C} z^{r+s-1} e^{-\lambda z}. \end{aligned}$$



例6.6. N 件产品中有 D 件次品. 随机抽 n 件, 包含 X 件次品. 求 EX 与 $\text{var}(X)$. (其中, $N \geq n \geq 2$).

- 随机数目的分解: $X = X_1 + \dots + X_n$, 其中

$$X_i = \begin{cases} 1, & \text{若第 } i \text{ 件是次品;} \\ 0, & \text{若第 } i \text{ 件是合格品.} \end{cases}$$

- 由期望的线性、伯努利分布的期望

$$EX = \sum_{i=1}^n EX_i = \sum_{i=1}^n P(\text{第 } i \text{ 件是次品}) = n \frac{D}{N}.$$

- $\text{var}(X) = EX^2 - (EX)^2$. 根据对称性,

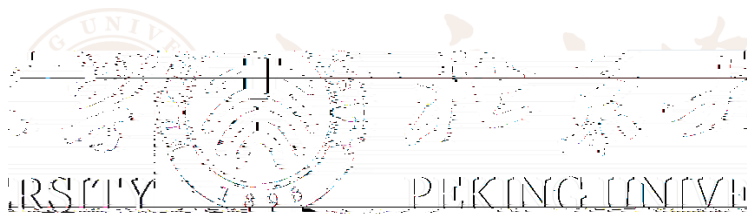
$$EX^2 = \sum_{i=1}^n EX_i^2 + \sum_{i \neq j} EX_i X_j = nEX_1^2 + n(n-1)EX_1 X_2,$$

- 由乘法公式,

$$EX_1 X_2 = P(\text{前两件都是次品}) = \frac{D}{N} \cdot \frac{D-1}{N-1}.$$

因此,

$$\begin{aligned} \text{var}(X) &= n \frac{D}{N} + n(n-1) \frac{D}{N} \cdot \frac{D-1}{N-1} - \left(n \frac{D}{N}\right)^2 \\ &= \frac{n(N-n)D(N-D)}{N^2(N-1)}. \end{aligned}$$



• 定理6.5. 设 $\xi = (X_1, \dots, X_n)$ 的协方差阵为 Σ ,

$Y_i = \sum_{j=1}^n a_{ij} X_j, j = 1, \dots, m$. 记 $\mathbf{A} = (a_{ij})_{m \times n}$,

则 $\eta = (Y_1, \dots, Y_m)$ 的协方差阵为 $\mathbf{A}\Sigma\mathbf{A}^T$.

• 定理6.6. 进一步, 若 $\xi \sim N(\bar{\mu}, \Sigma)$, 则 $\eta \sim N(\bar{\mu}\mathbf{A}^T, \mathbf{A}\Sigma\mathbf{A}^T)$.

• 设 X_1, \dots, X_n 独立同分布, 将它们从小到大排列:

$X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$

称 $X_{(k)}$ 为第 k 个顺序统计量.

例6.7. 设 X_1, \dots, X_n 独立同分布, 都服从 $U(0, 1)$.

求 $EX_{(k)}$ 与 $\text{var}(X_{(k)})$.

- 方法一、 $\forall 0 \leq x \leq 1$,

$$P(X_{(k)} \leq x) = \sum_{i=k}^n \frac{n!}{i!(n-i)!} x^i (1-x)^{n-i}.$$

$$\begin{aligned} &= \frac{n!}{i!(n-i)!} (ix^{i-1})(1-x)^{n-i} - x^i(n-i)(1-x)^{n-i-1} \\ &= \frac{n!}{(i-1)!(n-i)!} x^{i-1}(1-x)^{n-i} - \frac{n!}{i!(n-i)!} x^i(1-x)^{n-i-1} \\ &= a_{i-1} - a_i, \end{aligned}$$

- $i = n$ 时, $(x^n)' = a_{n-1}$, 于是, $\forall 0 \leq x \leq 1$,

$$p_{X_{(k)}}(x) = \sum_{i=k}^{n-1} (a_{i-1} - a_i) = nx^{n-1} = a_{k-1}.$$

- 已有 $q_{\mathbf{k}}(x) := p_{\mathbf{X}_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k}$.

- $\forall \ell, m \geq 1,$

$$\int_0^1 x(1-x)^m dx = \frac{1}{\ell+1} \int_0^1 (1-x)^m dx + 1$$

$$= \frac{1}{\ell+1} \int_0^1 x^{+1} d(1-x)^m = \frac{m}{\ell+1} \int_0^1 x^{+1} (1-x)^{m-1} dx$$

$$= \dots = \frac{m!}{(\ell+1) \dots (\ell+m)} \int_0^1 x^{+m} dx = \frac{\ell! m!}{(\ell+m-1)!}$$

期望: 取 $\ell = k, m = n - k$, 知

$$EX_{(k)} = \int_0^1 x q_{\mathbf{k}}(x) dx = \frac{n!}{(k-1)!(n-k)!} \int_0^1 x^k (1-x)^{n-k} dx$$

$$= \frac{n!}{(k-1)!(n-k)!} \cdot \frac{k!(n-k)!}{(n-1)!} = \frac{k}{n-1}.$$

- 已有 $q_{\mathbf{k}}(x) := p_{\mathbf{X}_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k}$.

$$\int_0^1 x(1-x)^m dx = \frac{m!}{(m+1)!}.$$

- 二阶矩: 取 $\ell = k-1, m = n-k$,

$$\begin{aligned} EX_{(k)}^2 &= \int_0^1 x^2 q_{\mathbf{k}}(x) dx = \frac{n!}{(k-1)!(n-k)!} \int_0^1 x^{k+1}(1-x)^{n-k} dx \\ &= \frac{n!}{(k-1)!(n-k)!} \cdot \frac{(k-1)!(n-k)!}{(n-2)!} = \frac{k(k-1)}{(n-1)(n-2)}. \end{aligned}$$

$$\begin{aligned} \text{var}(X_{(k)}) &= EX_{(k)}^2 - (EX_{(k)})^2 = \frac{k(k-1)}{(n-1)(n-2)} - \frac{k^2}{(n-1)^2} \\ &= \frac{k^2(n-1) - k(n-1)(n-2)}{(n-1)^2(n-2)} = \frac{k(n-1-k)}{(n-1)^2(n-2)}. \end{aligned}$$

- 方法二、记

$$Y_1 = X_{(1)}, \quad Y_k = X_{(k)} - X_{(k-1)}, \quad (2 \leq k \leq n), \quad Y_{n+1} = 1 - X_{(n)}.$$

- 不加证明地接受如下对称性*:

对任意 $1, \dots, n-1$ 的全排 i_1, \dots, i_{n+1} , 都有

(Y_1, \dots, Y_{n+1}) 与 $(Y_{i_1}, \dots, Y_{i_{n+1}})$ 同分布.

- 期望: 注意到 $S := Y_1 + \dots + Y_{n+1} = 1$, 故

$$1 = ES = (n-1)EY_k \quad EY_k = \frac{1}{n-1}.$$

- $X_{(k)} = Y_1 + \dots + Y_k$, 故

$$EX_{(k)} = \frac{k}{n-1}.$$

- Y_k 的方差: 记 $\sigma^2 := \text{var}(Y_k) = \text{var}(Y_{n+1}) = \text{var}(X_{(n)})$.

$$F_n(x) := P(X_{(n)} \leq x) = x^n \quad q_n(x) = nx^{n-1}, \quad 0 \leq x \leq 1$$

$$\begin{aligned} \sigma^2 &= \int_0^1 nx^{n+1} dx - \left(\int_0^1 nx^n dx \right)^2 \\ &= \frac{n}{n+2} - \frac{n^2}{(n+1)^2} = \frac{n}{(n+1)^2(n+2)}. \end{aligned}$$

- 协方差: 注意到 $S := Y_1 + \cdots + Y_{n+1} = 1$, 故 $\text{var}(S) = 0$.

- 记 $\sigma_{12} = \text{cov}(Y_1, Y_2)$.

$$\text{var}(S) = (n+1)\sigma^2 + (n+1)n\sigma_{12} = 0 \quad \sigma_{12} = -\frac{\sigma^2}{n+2}.$$

- $X_{(k)}$ 的方差:

$$\text{var}(X_{(k)}) = k\sigma^2 - k(k-1)\sigma_{12} = \frac{k(n-k+1)}{(n+1)^2(n+2)}.$$