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- 定义6.3. 若存在$p(x_1, \cdots, x_{\mathbf{n}})$ 使得对任意n 维矩形 D 都有

$$P(\xi \in D) = \int \cdots \int_{\mathbf{D}} p(x_1, \cdots, x_{\mathbf{n}}) dx_1 \cdots dx_{\mathbf{n}},$$

则称ξ 为连续型随机向量, 称$p(x_1, \cdots, x_{\mathbf{n}})$ 为ξ 的联合密度, 也记为$p_{\mathbf{X}_1, \ldots, \mathbf{X}_n}$. (注: ★★ 对一般的$D$ 都成立.)

- 定义6.4. 对任意1 (, x ★

例6.1 (项分布). 设$U_1,\cdots,U_n$ 相互独立, 都服从如下分布:

$$P(U_i = k) = p_k, \quad k = 1,\cdots,t,$$

其中$t \quad 2$, $p_k \quad 0, \forall k$ 且$p_1 \quad \cdots \quad p_t = 1$.

- 背景模型: n 次独立重复试验(投掷一枚t 面骰子). 记

$$X_k = |\{1 \quad i \quad n : U_i = k\}| = \sum_{i=1}^{n} 1_{\{U_i=k\}}.$$

- $\xi = (X_1,\cdots,X_t)$ 的联合分布列:

$$P(\xi = (i_1,\cdots,i_t)) = \frac{n!}{i_1!\cdots i_t!} p_1^{i_1}\cdots p_t^{i_t}.$$

- 因为$X_t = n \quad \sum_{s=1}^{t-1} X_s$, 所以$\xi$ 与$(X_1,\cdots,X_{t-1})$ 等价.
- 任意边缘都服从 项分布.

  例, (X_1,X_2) 服从三项分布; 特别地, X_k 服从二项分布:

$$\text{若}U_i = k, \text{则令}V_i = 1; \text{若}U_i \quad k, \text{则令}V_i = 0.$$

- 定义6.5. 若对任意$a_i \quad b_i$, $i=1,\cdots,n$ 都有

$$P(a_1 \quad X_1 \quad b_1, \cdots, a_n \quad X_n \quad b_n)$$
$$=P(a_1 \quad X_1 \quad b_1)\cdots P(a_n \quad X_n \quad b_n),$$

则称$X_1,\cdots,X_n$相互独立.

- 若$X_1,\cdots,X_n$ 相互独立, 且$F_{X_i}=F_{X_1}$, $i=2,\cdots,n$, 则称$X_1,\cdots,X_n$ 独立同分布.
- 若相互独立, 则上式中的$a_i \quad X_i \quad b_i$ 可以改为$X_i \in B_i$.

- 相互独立的充要条件与充分条件:

$$F_{X_1,\cdots,X_n}(x_1,\cdots,x_n) = F_{X_1}(x_1)\cdots F_{X_n}(x_n) = f_1(x_1)\cdots f_n(x_n).$$

- 离散型:

$$P\left(X_1 = x_{i_1}^{(1)},\cdots,X_n = x_{i_n}^{(n)}\right) = P\left(X_1 = x_{i_1}^{(1)}\right)\cdots P\left(X_n = x_{i_n}^{(n)}\right) = p_{i_1}^{(1)}\cdots p_{i_n}^{(n)}.$$

- 连续型(定理6.1):

$$p_{X_1,\cdots,X_n}(x_1,\cdots,x_n) = p_{X_1}(x_1)\cdots p_{X_n}(x_n) = p_1(x_1)\cdots p_n(x_n).$$

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- 定义6.6. 称$(EX_1, \cdots, EX_{\mathbf{n}})$ 为ξ 的期望, 记为$E\xi$.
- 定义6.7. 记$\sigma_{\mathbf{ij}} = \mathrm{cov}(X_{\mathbf{i}}, X_{\mathbf{j}})$, $\rho_{\mathbf{ij}} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}$.
  称$\mathbf{\Sigma} = (\sigma_{\mathbf{ij}})_{\mathbf{n}\times\mathbf{n}}$, $\mathbf{R} = (\rho_{\mathbf{ij}})_{\mathbf{n}\times\mathbf{n}}$ 为ξ 的协方差阵, 相关系数阵.
- 定义6.8. n 维正态分布. 假设ξ 有如下的联合密度, 则称ξ 服从n 维正态分布, 记为$\xi \quad N(\vec{\mu}, \mathbf{\Sigma})$.

$$p(\vec{x}) = \frac{1}{\sqrt{2\pi}^{\mathbf{n}}\sqrt{|\mathbf{\Sigma}|}} \exp\left\{ \quad \frac{1}{2}(\vec{x} \quad \vec{\mu})\mathbf{\Sigma}^{-1}(\vec{x} \quad \vec{\mu})^{\mathbf{T}} \right\}.$$

- $n = 1$ 与$n = 2$ 的特例已介绍.
- $N(\vec{\mu}, \mathbf{\Sigma})$ 的数字特征: $\mu_{\mathbf{i}} = EX_{\mathbf{i}}$, $\sigma_{\mathbf{ij}} = \mathrm{cov}(X_{\mathbf{i}}, X_{\mathbf{j}})$.
- $X_1, \cdots, X_{\mathbf{n}}$ 相互独立当且仅当$\sigma_{\mathbf{ij}} = 0$, $\forall i \quad j$.
- 边缘分布, 条件分布都是正态.

3. n 个随机变量的函数Y f $X_1,$ $, X_n$

例6.3, 6.4, 定义6.9. 若X 与Y 独立, $X \quad \Gamma(r,\lambda)$, $Y \quad \Gamma(s,\lambda)$. 则

$$X \quad Y \quad \Gamma(r \quad s,\lambda).$$

- 密度:

$$p_{\mathbf{X}}(x) = \frac{\lambda^{\mathbf{r}}}{\Gamma(r)} x^{\mathbf{r}-1} e^{-\mathbf{\lambda x}}, \quad x \quad 0.$$

- $Z = X \quad Y$: $p_{\mathbf{Z}}(z) = \int p_{\mathbf{X}}(x) p_{\mathbf{Y}}(z \quad x)dx$. $\forall z \quad 0$,

$$p_{\mathbf{Z}}(z) = C \int_0^{\mathbf{z}} x^{\mathbf{r}-1} e^{-\mathbf{\lambda x}} \cdot (z \quad x)^{\mathbf{s}-1} e^{-\mathbf{\lambda(z-x)}} dx$$

$$= C e^{-\mathbf{\lambda z}} \int_0^1 (t{\color{red}z})^{\mathbf{r}-1} ((1 \quad t){\color{red}z})^{\mathbf{s}-1} d(t{\color{red}z}) = \hat{C} {\color{red}z^{\mathbf{r}+\mathbf{s}-1}} e^{-\mathbf{\lambda z}}.$$

例6.6. N 件产品中有D 件次品. 随机抽n 件, 包含X 件次品. 求EX 与var(X). (其中, $N \quad n \quad 2$).

- 随机数目的分解: $X = X_1 \quad \cdots \quad X_{\mathbf{n}}$, 其中

$$X_{\mathbf{i}} = \begin{cases} 1, & \text{若第}i\text{ 件是次品;} \\ 0, & \text{若第}i\text{ 件是合格品.} \end{cases}$$

- 由期望的线性、伯努利分布的期望,

$$EX = \sum_{\mathbf{i}=1}^{\mathbf{n}} EX_{\mathbf{i}} = \sum_{\mathbf{i}=1}^{\mathbf{n}} P(\text{第}i\text{ 件是次品}) = n\frac{D}{N}.$$

- $\mathrm{var}(X) = EX^2 \quad (EX)^2$. 根据对称性,

$$EX^2 = \sum_{i=1}^{n} EX_i^2 \quad \sum_{i\neq j} EX_iX_j = nEX_1^2 \quad n(n \quad 1)EX_1X_2,$$

- 由乘法公式,

$$EX_1X_2 = P(\text{前两件都是次品}) = \frac{D}{N}\cdot\frac{D \quad 1}{N \quad 1}.$$

- 因此,

$$\mathrm{var}(X) = n\frac{D}{N} \quad n(n \quad 1)\frac{D}{N}\cdot\frac{D \quad 1}{N \quad 1} \quad \left(n\frac{D}{N}\right)^2$$

$$= \frac{n(N \quad n)D(N \quad D)}{N^2(N \quad 1)}.$$

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- 定理6.5. 设$\xi=(X_1,\cdots,X_{\mathsf{n}})$ 的协方差阵为$\mathbf{\Sigma}$, $Y_{\mathsf{i}}=\sum_{\mathsf{j}=1}^{\mathsf{n}} a_{\mathsf{ij}}X_{\mathsf{j}}$, $j=1,\cdots,m$. 记$\mathbf{A}=(a_{\mathsf{ij}})_{\mathsf{m}\times\mathsf{n}}$, 则$\eta=(Y_1,\cdots,Y_{\mathsf{m}})$ 的协方差阵为$\mathbf{A\Sigma A}^{\mathsf{T}}$.
- 定理6.6. 进一步, 若$\xi\sim N(\vec{\mu},\mathbf{\Sigma})$, 则$\eta\sim N(\vec{\mu}\mathbf{A}^{\mathsf{T}},\mathbf{A\Sigma A}^{\mathsf{T}})$.
- 设$X_1,\cdots,X_{\mathsf{n}}$ 独立同分布, 将它们从小到大排列:

$$X_{(1)}\le X_{(2)}\le\cdots\le X_{(\mathsf{n})},$$

称$X_{(\mathsf{k})}$ 为第k 个顺序统计量.

例6.7. 设$X_1, \cdots, X_{\mathbf{n}}$ 独立同分布, 都服从$U(0,1)$.
求$EX_{(\mathbf{k})}$ 与$\text{var}(X_{(\mathbf{k})})$.

- 方法一、$\forall 0 \quad x \quad 1$,

$$P(X_{(\mathbf{k})} \quad x) = \sum_{\mathbf{i}=\mathbf{k}}^{\mathbf{n}} \frac{n!}{i!(n \quad i)!} x^{\mathbf{i}}(1 \quad x)^{\mathbf{n}-\mathbf{i}}.$$

- $k \quad i \quad n \quad 1$, $\star\star'$

$$= \frac{n!}{i!(n \quad i)!}(ix^{\mathbf{i}-1}(1 \quad x)^{\mathbf{n}-\mathbf{i}} \quad x^{\mathbf{i}}(n \quad i)(1 \quad x)^{\mathbf{n}-\mathbf{i}-1})$$

$$= \frac{n!}{(i \quad 1)!(n \quad i)!} x^{\mathbf{i}-1}(1 \quad x)^{\mathbf{n}-\mathbf{i}} \quad \frac{n!}{i!(n \quad i \quad 1)!} x^{\mathbf{i}}(1 \quad x)^{\mathbf{n}-\mathbf{i}-1}$$

$$= a_{\mathbf{i}-1} \quad a_{\mathbf{i}},$$

- $i = n$ 时, $(x^{\mathbf{n}})' = a_{\mathbf{n}-1}$, 于是, $\forall 0 \quad x \quad 1$,

$$p_{\mathbf{X}_{(k)}}(x) = \sum_{\mathbf{i}=\mathbf{k}}^{\mathbf{n}-1} (a_{\mathbf{i}-1} \quad a_{\mathbf{i}}) \quad nx^{\mathbf{n}-1} = a_{\mathbf{k}-1}.$$

- 已有$q_{\mathbf{k}}(x) := p_{\mathbf{X}_{(k)}}(x) = \frac{\mathbf{n}!}{(\mathbf{k}-1)!(\mathbf{n}-\mathbf{k})!} x^{\mathbf{k}-1}(1-x)^{\mathbf{n}-\mathbf{k}}$.
- $\forall \ell, m > -1$,

$$\int_0^1 x^{\ell}(1-x)^{\mathbf{m}}dx = \frac{1}{\ell+1}\int_0^1 (1-x)^{\mathbf{m}}dx^{\ell+1}$$

$$= \frac{1}{\ell+1}\int_0^1 x^{\ell+1}d(1-x)^{\mathbf{m}} = \frac{m}{\ell+1}\int_0^1 x^{\ell+1}(1-x)^{\mathbf{m}-1}dx$$

$$= \cdots = \frac{m!}{(\ell+1)\cdots(\ell+m)}\int_0^1 x^{\ell+\mathbf{m}}dx = \frac{\ell!m!}{(\ell+m+1)!}.$$

- 期望：取$\ell = k, m = n-k$，知

$$EX_{(\mathbf{k})} = \int_0^1 x q_{\mathbf{k}}(x)dx = \frac{n!}{(k-1)!(n-k)!}\int_0^1 x^{\mathbf{k}}(1-x)^{\mathbf{n}-\mathbf{k}}dx$$

$$= \frac{n!}{(k-1)!(n-k)!}\cdot\frac{k!(n-k)!}{(n+1)!} = \frac{k}{n+1}.$$

- 已有$q_k(x) := p_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k}$.

$$\int_0^1 x^\ell (1-x)^m dx = \frac{\ell! m!}{(\ell+m+1)!}.$$

- 二阶矩: 取$\ell = k+1$, $m = n-k$,

$$EX_{(k)}^2 = \int_0^1 x^2 q_k(x)dx = \frac{n!}{(k-1)!(n-k)!}\int_0^1 x^{k+1}(1-x)^{n-k}dx$$
$$= \frac{n!}{(k-1)!(n-k)!}\cdot\frac{(k+1)!(n-k)!}{(n+2)!} = \frac{k(k+1)}{(n+1)(n+2)}.$$

- 方差:

$$\mathrm{var}(X_{(k)}) = EX_{(k)}^2 - \left(EX_{(k)}\right)^2 = \frac{k(k+1)}{(n+1)(n+2)} - \frac{k^2}{(n+1)^2}$$
$$= \frac{k^2(n+1) + k(n+1) - k^2(n+2)}{(n+1)^2(n+2)} = \frac{k(n+1-k)}{(n+1)^2(n+2)}.$$

- 方法二、记

$$Y_1 = X_{(1)}, \quad Y_{\mathbf{k}} = X_{(\mathbf{k})} \quad X_{(\mathbf{k}-1)}, (2 \quad k \quad n), \quad Y_{\mathbf{n}+1} = 1 \quad X_{(\mathbf{n})}.$$

- 不加证明地接受如下对称性*:

对任意$1, \cdots, n \quad 1$ 的全排$i_1, \cdots, i_{\mathbf{n}+1}$, 都有

$$(Y_1, \cdots, Y_{\mathbf{n}+1}) \text{ 与 } (Y_{\mathbf{i}_1}, \cdots, Y_{\mathbf{i}_{n+1}}) \text{ 同分布.}$$

- 期望：注意到$S := Y_1 \quad \cdots \quad Y_{\mathbf{n}+1} = 1$, 故

$$1 = ES = (n \quad 1) EY_{\mathbf{k}} \quad EY_{\mathbf{k}} = \frac{1}{n \quad 1}.$$

- $X_{(\mathbf{k})} = Y_1 \quad \cdots \quad Y_{\mathbf{k}}$, 故

$$EX_{(\mathbf{k})} = \frac{k}{n \quad 1}.$$

- $Y_{\mathbf{k}}$ 的方差：记$\sigma^2 := \mathrm{var}(Y_{\mathbf{k}}) = \mathrm{var}(Y_{\mathbf{n}+1}) = \mathrm{var}(X_{(\mathbf{n})})$.

$$F_{\mathbf{n}}(x) := P(X_{(\mathbf{n})} \quad x) = x^{\mathbf{n}} \quad q_{\mathbf{n}}(x) = nx^{\mathbf{n}-1},\ 0 \quad x \quad 1$$

$$\sigma^2 = \int_0^1 nx^{\mathbf{n}+1}dx \quad \left(\int_0^1 nx^{\mathbf{n}}dx\right)^2$$

$$= \frac{n}{n \quad 2} \quad \frac{n^2}{(n \quad 1)^2} = \frac{n}{(n \quad 1)^2(n \quad 2)}.$$

- 协方差：注意到$S := Y_1 \quad \cdots \quad Y_{\mathbf{n}+1} = 1$，故$\mathrm{var}(S) = 0$.
- 记$\sigma_{12} = \mathrm{cov}(Y_1, Y_2)$.

$$\mathrm{var}(S) = (n \quad 1)\sigma^2 \quad (n \quad 1)n\sigma_{12} \quad \sigma_{12} = \frac{1}{(n \quad 1)^2(n \quad 2)}.$$

- $X_{(\mathbf{k})}$ 的方差：

$$\mathrm{var}(X_{\mathbf{k}}) = k\sigma^2 \quad k(k \quad 1)\sigma_{12} = \frac{k(n \quad 1 \quad k)}{(n \quad 1)^2(n \quad 2)}.$$