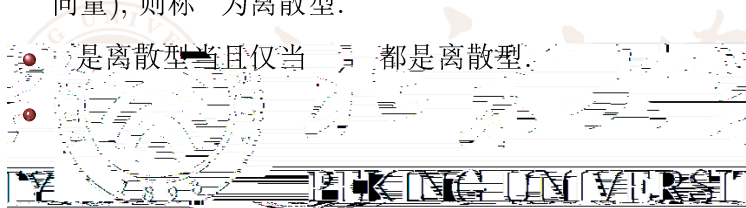


1. 离散型情形

§3.2 二维随机向量的联合分布与边缘分布, §3.7 条件分布

- 定义2.1 & 2.2. 若 (X, Y) 取有限个或可列个“值”(二维向量), 则称 (X, Y) 为离散型.

- (X, Y) 是离散型当且仅当 X 和 Y 都是离散型.



- 定义2.3. 设 (X, Y) , 则 X 的分布称为 Y 关于 X 的边缘分布. 关于 Y 的边缘分布类似.

- 例2.5. $0 \leq \rho \leq \frac{1}{4}$,

$$P(X=0, Y=0) = P(X=1, Y=1) = \frac{1}{4} + \rho;$$

$$P(X=0, Y=1) = P(X=1, Y=0) = \frac{1}{4} - \rho.$$

总有, $P(X=1, Y=1) = (1 - \frac{1}{2}) \cdot \frac{1}{2}$.

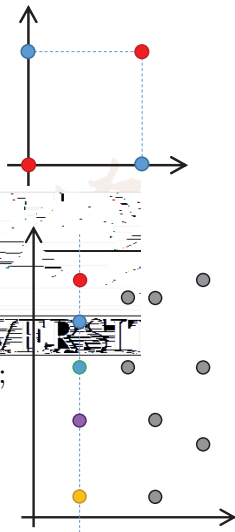
给定 Y 将

$$P(X=0 | Y=j) = \frac{P(X=0, Y=j)}{P(Y=j)} = \frac{1}{2} - \rho$$

称为在 $Y=j$ 的条件下, X 的条件分布(列);

$P(X=1 | Y=j)$ 的条件分布类似. (7.3)

- 联合分布列 \Leftrightarrow 边缘分布列、条件分布列.



例2.2 & 2.3: 有大量粉笔, 含白、黄、红三种颜色, 比例分别为 $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{3}$. 从中抽取 n 支. 求: 恰好抽到 X_1 支白, X_2 支黄的概 率.



- $(X_1 = 1, X_2 = 2) = \frac{k_1}{n} \frac{k_2}{n-k_1} \frac{k_1}{1} \frac{k_2}{2} \frac{n-k_1-k_2}{3}$.

- 的边缘分布:

$$P(X_1 = 1) = \sum_{k_2=0}^{n-k_1} \frac{k_1}{n} \frac{k_2}{n-k_1} \frac{k_1}{1} \frac{k_2}{2} \frac{n-k_1-k_2}{3}$$

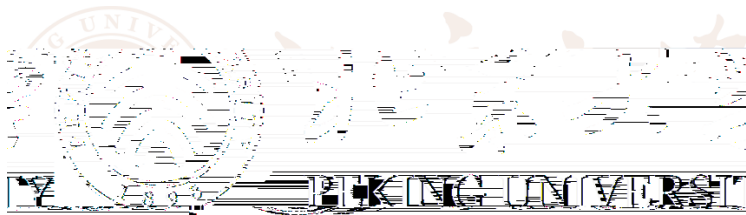
$$= \frac{k_1}{n} \frac{k_1}{1} (1 - \frac{1}{1})^{n-k_1}$$

的条件分布: 给定 $X_1 = 1$,

$$P(X_2 = 2 | X_1 = 1) = \frac{P(X_1 = 1, X_2 = 2)}{P(X_1 = 1)} = \frac{\frac{k_2}{n-k_1} \left(\frac{2}{2+k_3}\right)^{k_2} \left(\frac{3}{2+k_3}\right)^{k_3}}{\frac{k_1}{n}}$$

$$= \frac{k_2}{n-k_1} \left(\frac{2}{2+k_3}\right)^{k_2} \left(\frac{3}{2+k_3}\right)^{k_3} \quad k_2 = 0, 1, \dots, n-1$$

2. 连续型情形



- 定理2.1. 若 (X, Y) 是连续型, 则 X, Y 都是连续型, 且

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx$$

- 称 $f_X(\cdot)$ 与 $f_Y(\cdot)$ 为 (X, Y) 的边缘密度.

- 给定 y , 满足 $f_Y(y) > 0$. 称(关于 X 的函数)

为在 $Y=y$ 的条件下, X 的条件密度. (7.5)

- 联合密度 \Leftrightarrow 边缘密度、条件密度.

- 定义2.5. 假设 A 是 \mathbb{R}^2 中面积为 a 的区域. 若

$$f(x, y) = \frac{A \text{ 的面积}}{a} \quad \forall \text{子区域 } A$$

则称 (X, Y) 服从 A 上的均匀分布, 记为 $(X, Y) \sim U(A)$.

- 联合密度: $f(x, y) = \frac{1}{a}, (x, y) \in A$.

- 边缘密度:

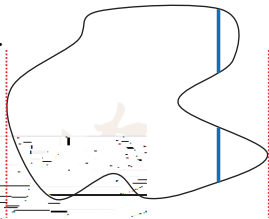
$$f_X(x) = \frac{|G_{2,x}|}{a}, \quad x \in I$$

其中, $I = \{x : (x, y) \in A\}$, $|G_{2,x}|$ 为其总长度;

- 条件密度: $f_{Y|X}(y|x) = \frac{1}{|G_{2,x}|}, (x, y) \in A$.

- $f_{Y|X}(y|x)$ 就是**固定**, 将 (x, y) 视为 A 的函数**归一化**,

$$f_{Y|X}(y|x) = \frac{1}{|G_{2,x}|} \mathbb{1}_A(x, y).$$



例2.7. 为由 $x = y^2$ 和 $x = 1$ 所围成的有限区域. $(X, Y) \sim (\quad)$.

求: (X, Y) 的联合密度与边缘密度.

- 的面积: $A = \int_0^1 (1 - y^2) dy = \frac{1}{6}$.

- 联合密度: $f(x, y) = 6, (x, y) \in D$.

- 边缘密度:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_{-\sqrt{x}}^{\sqrt{x}} 6 dy = 6(\sqrt{x} - (-\sqrt{x})) = 12\sqrt{x}, 0 \leq x \leq 1.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_{y^2}^1 6 dx = 6(1 - y^2), 0 \leq y \leq 1.$$

- 条件密度: 固定 $y \in (0, 1)$,

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{y}-y}, \sqrt{y} \leq x \leq 1.$$

- 注: f_X, f_Y 都取遍 $(0, 1)$, 但 $f_{X|Y}$ 不能取遍 $(0, 1) \times (0, 1)$.

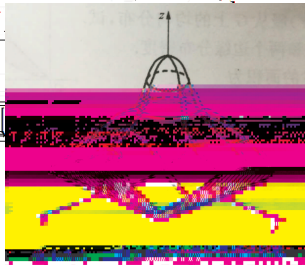
- 定义2.6, 例2.8 & 例7.5. 若 (X, Y) 的联合密度 $f(x, y)$ 有如下表达式, 则称 (X, Y) 服从二维(元)正态分布.

$$\frac{1}{2\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{z_1^2 + z_2^2 - 2\rho z_1 z_2}{2(1-\rho^2)}\right\}$$

其中,

有5个参数:

$$\begin{aligned} \sigma_1 > 0, \\ \sigma_2 > 0, \\ \rho \in (-1, 1). \end{aligned}$$



• 联合密度: $f(x, y) = \frac{1}{2\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{x-\mu_1}{\sigma_1} - \rho\frac{y-\mu_2}{\sigma_2}\right]^2 - \frac{1}{2(1-\rho^2)}\left[\frac{y-\mu_2}{\sigma_2} + \rho\frac{x-\mu_1}{\sigma_1}\right]^2\right\}$

$$\frac{1}{2\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{x-\mu_1}{\sigma_1} - \rho\frac{y-\mu_2}{\sigma_2}\right]^2 - \frac{1}{2(1-\rho^2)}\left[\frac{y-\mu_2}{\sigma_2} + \rho\frac{x-\mu_1}{\sigma_1}\right]^2\right\}$$

• 边缘密度: $f_1(x) \sim N(\mu_1, \sigma_1^2)$, $f_2(y) \sim N(\mu_2, \sigma_2^2)$. 例如,

$$f_1(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_{-\infty}^{\infty} \frac{1}{2\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{x-\mu_1}{\sigma_1} - \rho\frac{y-\mu_2}{\sigma_2}\right]^2 - \frac{1}{2(1-\rho^2)}\left[\frac{y-\mu_2}{\sigma_2} + \rho\frac{x-\mu_1}{\sigma_1}\right]^2\right\} dy$$

$$= \frac{1}{2\sigma_1\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(1-\rho^2)u^2}{2(1-\rho^2)}\right\} \exp\left\{-\frac{(\rho(x-\mu_1) + y - \mu_2)^2}{2(1-\rho^2)}\right\} dy$$

$$= \frac{1}{2\sigma_1\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}u^2 - \frac{(\rho(x-\mu_1) + y - \mu_2)^2}{2(1-\rho^2)}\right\} dy = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}$$

• 联合密度: $f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{\left(\frac{x-\mu_1}{\sigma_1} - \rho\frac{y-\mu_2}{\sigma_2}\right)^2 + \frac{(1-\rho^2)}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right\}$

$$\exp\left\{-\frac{\left(\frac{x-\mu_1}{\sigma_1} - \rho\frac{y-\mu_2}{\sigma_2}\right)^2 + \frac{(1-\rho^2)}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right\}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{\left(\frac{x-\mu_1}{\sigma_1} - \rho\frac{y-\mu_2}{\sigma_2}\right)^2 + \frac{(1-\rho^2)}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right\}$$

• 边缘密度:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}$$

• 条件密度:

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{\left(\frac{y-\mu_2}{\sigma_2} - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right)^2}{2(1-\rho^2)}\right\}$$

• 另解:

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{\left(\frac{y-\mu_2}{\sigma_2} - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right)^2}{2(1-\rho^2)}\right\}$$

例2.9. $X = (X, Y)$ 与 $Z = (X, Y)$ 分别有联合密度

$$f(x, y) = \frac{1}{2} e^{-\frac{x^2+y^2}{2}} \quad g(x, y) = 2 f(x, y) \quad x, y > 0.$$

- 服从二维正态分布, $(X, Y) \sim (0, 1)$, $\rho = 0$.
- 不服从二维正态分布.

• 但 $(X, Y) \sim (0, 1)$. 例如, $\forall u > 0$,

$$\begin{aligned} P(X > u) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2 f(x, y) dx dy \\ &= \int_0^{\infty} \frac{1}{2} e^{-\frac{u^2+v^2}{2}} dv = \frac{1}{2} e^{-\frac{u^2}{2}} \int_0^{\infty} e^{-\frac{v^2}{2}} dv \\ &= \frac{1}{2} e^{-\frac{u^2}{2}} \times \frac{1}{2} \sqrt{2} = \frac{1}{\sqrt{2}} e^{-\frac{u^2}{2}} \end{aligned}$$

- 注: X, Y 都是正态变量, 不能推出 (X, Y) 是二维正态向量.

3. 一般情形

- 定义2.7. 称 $F(x, y) = P(X \leq x, Y \leq y)$ 为 (X, Y) 的联合分布函数, 也记为 $F_{X,Y}(x, y)$.

- 联合分布函数的性质: “单调”、“规范”、右连续,

$$F(x_2, y_2) - F(x_1, y_2) = F(x_2, y_1) - F(x_1, y_1) \geq 0,$$

$$\lim_{x \rightarrow -\infty} F(x, y) = \lim_{y \rightarrow -\infty} F(x, y) = 0, \quad \lim_{y \rightarrow +\infty} F(x, y) = 1.$$

- 连续型向量:

$$F(x, y) = \int_{-\infty}^x \int_{-\infty}^y f(u, v) \, du \, dv, \dots \Rightarrow f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y).$$