§2.6 随机变量的数学期望

期望(expectation)的含义: 均值(mean).

- X 的大量独立观测值(记为$a_1, a_2, \cdots, a_n$) 的算术平均:

$$\bar{a} = \frac{1}{n}(a_1 + \cdots + a_n).$$

- X 的所有可能值的加权平均(总和).

  例, $P(X = x_k) = p_k$, $k = 1, \cdots$.

  记$n_k = \#\{m : 1 \leqslant m \leqslant n, a_m = x_k\}$. 那么, 根据概率的频率含义, $\frac{n_k}{n} \approx p_k$, 于是

$$\bar{a} = \frac{1}{n}\sum_{k=1}^{K} n_k \approx \sum_{k=1}^{K} x_k p_k.$$

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(2) 二项分布.

$$(X = \ ) = C_n^k p^k q^{n-k} =: \ (n; \ ), \quad = 0, 1, \cdots, n, (q = 1 - p)$$

- $\forall 1 \leqslant \ \leqslant n$,

$$\cdot \ (n; \ ) = \ \cdot \frac{n!}{!(n - \ )!} p^k q^{n-k} = \frac{n!}{(\ -1)!(n - \ )!} p^k q^{n-k}$$

$$\frac{n \cdot (n-1)!}{(\ -1)!(n - \ )!} p \cdot p^{k-1} q^{n-k} = np \cdot \ (n-1, \ -1)$$

- 因此,

$$EX = \sum_{k=0}^{n} \ \cdot \ (n; \ ) = \sum_{k=1}^{n} np \cdot \ (n-1; \ -1)$$

$$= np \sum_{\ell=0}^{n-1} \ (n-1, \ ) = np$$

(7) 超几何分布.

$$(X = \ ) = \frac{C_D^k C_{N-D}^{n-k}}{C_N^n}, \qquad = 0, 1, \cdots, n,$$

- 记 $(\ , D, n; \ ) = A_1 \cdot A_2 \cdot A_3 =$

$$\frac{D!}{!(D - \ )!} \cdot \frac{(\ - D)!}{(n - \ )!(\ - D - (n - \ ))!} \cdot \frac{n!(\ - n)!}{!},$$

- 记 $x' = x - 1$, 则, $\forall 1 \le \ \le n$,

$$A_1 = \frac{D!}{(\ - 1)!(D - \ )!} = D \times \frac{D'!}{(\ ')!(D' - \ ')!},$$

- 进一步,

$$A_2 = \frac{(\ ' - D')!}{(n' - \ ')!(\ ' - D' - (n' - \ '))!},$$

$$A_3 = \frac{n \cdot n'!(\ ' - n')!}{\cdot \ '!} = \frac{n}{\ } \times \frac{n'!(\ ' - n')!}{\ '!},$$

- 记$x' = x - 1$. 则$\forall 1 \leqslant \quad \leqslant n$,

$$\cdot P(\ \ ..D..n;\ \ ) = \frac{nD}{\ } \times P(\ \ '..D'..n';\ \ ')$$

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(4) 几何分布.

$$P(X = k) = q^{k-1}p =: p_k, \quad k = 1, 2, \cdots, (q = 1 - p).$$

- 直接计算:

$$EX = \sum_{k=1}^{\infty} k p_k = \sum_{k=1}^{\infty} {\color{red}\sum_{\ell=1}^{k}} p_k = \sum_{\ell=1}^{\infty} {\color{cyan}\sum_{k=\ell}^{\infty} p_k}$$

$$= \sum_{\ell=1}^{\infty} p \cdot \frac{q^{\color{red}\ell-1}}{1-q} = \sum_{m=0}^{\infty} q^{\color{red}m} = \frac{1}{1-q} = \frac{1}{p}.$$

- 习题二、18. 若X取非负整数, 则$EX = \sum_{\ell=1}^{\infty} P(X \geqslant \ell)$.
- 证: ${\color{cyan}\sum_{k=\ell}^{\infty} p_k} = P(X \geqslant \ell)$.

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(2) 指数分布.

$$p(x) = \quad {}^{-\lambda x}, \quad x > 0,$$

- $\int_0^\infty x \cdot \quad {}^{-\lambda x} \quad x = -\int_0^\infty x \quad {}^{-\lambda x} = \int_0^\infty \quad {}^{-\lambda x} \quad x = \frac{1}{\lambda}.$
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(3) 正态分布.

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

- $X \sim N(0,1)$:

$$EX = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0,$$

- 同理, $X \sim N(\mu,\sigma^2)$, 则$p(\mu + x) = p(\mu - x)$, 因此$EX = \mu$.
- 例, 柯西分布,

$$p(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2},$$

但是, $\int_{-\infty}^{\infty} |x| p(x) dx = \infty$. 因此, EX 不存在!

(4) 伽玛分布.

$$p(x) = \frac{\alpha}{\Gamma(\ )} x^{\alpha-1} \quad {}^{-\beta x}, \quad x > 0,$$

- $\forall x > 0,$

$$xp(x) = \frac{\alpha}{\Gamma(\ )} x^{\alpha} \quad {}^{-\beta x} = \frac{\Gamma(\ +1)}{\Gamma(\ )} \cdot \frac{\alpha+1}{\Gamma(\ +1)} x^{\alpha} \quad {}^{-\beta x} = — \cdot \hat{p}(x),$$

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- 推论6.2. (1) 线性: 假设EX, EY 存在. 则,

$$E(aX + \ Y) = aEX + \ EY.$$

- 推论6.2. (2) 和的期望: 假设$EX_1, \cdots, EX_n$ 都存在, $\xi = X_1 + \cdots + X_n$. 则$E\xi$存在, 且

$$E\xi = EX_1 + \cdots + EX_n.$$

- 例. 超几何分布$\xi \sim H(\ , D, n)$.
  若第 个产品是次品, 则令$X_i = 1$; 否则, 令$X_i = 0$. 则,

$$\xi = X_1 + \cdots + X_n \Rightarrow E\xi = np.$$

- 定理6.4. (马尔可夫不等式). 设$X \geqslant 0$, 且EX 存在. 则对任意$C>0$, 有

$$P(X \geqslant C) \leqslant \frac{1}{C} EX.$$

- 证: 令$A=\{X \geqslant C\}$. 则$1_A \leqslant \frac{X}{C}$. 于是,

$$P(A) = E1_A \leqslant E\frac{X}{C} = \frac{1}{C} EX.$$

- 例. 若$X \geqslant 0$, 且$EX=0$, 则

$$P\left(X \geqslant \frac{1}{n}\right) \leqslant nEX = 0$$

$$\Rightarrow P(X>0) = \lim_{n\to\infty} P\left(X \geqslant \frac{1}{n}\right) = 0.$$

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