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m , 2009

$(X_1, Y_1), \ldots, (X_n, Y_n)$,

$$Y = g(X) + \varepsilon, \tag{1}$$

X $L_2(\mathcal{I})$ $\mathcal{I} = [0, 1]$, g $L_2(\mathcal{I})$ ε X, g g $g(x) = a + bx$, a b, b $n^{-1/2}$, (1991), (2002), (2003), (2003), (2003), (2005), (2005), (2005) (2007).

X g (1) (1998) (2002) (2003, 2004, 2006) (2007).

$L_2(\mathcal{I})$, $\psi_1, \psi_2, \ldots$, $V(s,t) = \{X(s), X(t)\}$:

$$V(s,t) = \sum_{j=1} \theta_j \psi_j(u) \psi_j(v), \tag{2}$$

ψ_j' θ_j' V. (2)

$\theta_1 \geq \theta_2 \geq \cdots$. [illegible] ψ_j' [illegible] θ_j' [illegible] $\hat{V}$ [illegible] V,

$$(3) \qquad \hat{V}(s,t) = \frac{1}{n} \sum_{i=1}^{n} \{X_i(s) - \bar{X}(s)\}\{X_i(t) - \bar{X}(t)\} = \sum_{j=1}^{\infty} \hat{\theta}_j \hat{\psi}_j(s) \hat{\psi}_j(t),$$

[illegible] $\bar{X} = n^{-1} \sum_i X_i$ [illegible] $\hat{\theta}_1 \geq \hat{\theta}_2 \geq \cdots$. [illegible] $\hat{\theta}_j$ [illegible] $j \geq n+1$, [illegible] $\hat{\psi}_{n+1}, \hat{\psi}_{n+2}, \ldots$ [illegible]

[illegible] X [illegible]

$$(4) \qquad X - E(X) = \sum_{j=1}^{\infty} \xi_j \psi_j,$$

[illegible] $\xi_j = \int_{\mathcal{I}} X \psi_j$ [illegible] θ_j. [illegible] $\hat{\psi}_j$ [illegible] ψ_j.

[illegible] 2, [illegible] g [illegible] (1) [illegible] ψ_j. [illegible] 3, [illegible] $\hat{g}$ [illegible] g [illegible] 1 [illegible] 2). [illegible] 3) [illegible] A [illegible] B [illegible] 4, [illegible] 5.

2. P[illegible] ed[illegible] e t[illegible] at[illegible] ced[illegible] e.

$$\hat{g}(x) = \frac{\sum_i Y_i K_i(x)}{\sum_i K_i(x)},$$

[illegible] $K_i(x) = K(\|x - X_i\|/h)$, K [illegible] h [illegible] $\|\cdot\|$ [illegible] L^2 [illegible] (2006) [illegible] g. [illegible] (1), [illegible] X [illegible] g. [illegible] 1 [illegible] 2 [illegible] 4 [illegible] X [illegible] 3 [illegible]

[illegible] (7).

[illegible]

A[illegible] 1. [illegible] K [illegible] $[0, c]$, [illegible] $c > 0$, [illegible] K [illegible] $[0, c]$.

[illegible] g [illegible] x [illegible] g_x [illegible] y [illegible] δ,

$$g(x + \delta y) = g(x) + \delta g_x y + o(\delta)$$

[illegible] $\delta \to 0$. [illegible]

(5) $$g_x = \sum_{j=1}^{\infty} \gamma_{xj} t_j,$$

[illegible] $\gamma_{xj} = g_x \psi_j$ [illegible], [illegible] t_j [illegible] y [illegible] $y_j = t_j(y) = y\psi_j$. [illegible] γ_{xj} [illegible] g_x [illegible] ψ_j.

[illegible] g_x, [illegible] γ_{xj}, [illegible] $a_x^m = (a_{x1}^m, a_{x2}^m, \ldots)$ [illegible] $a_x^m = (a_{x1}^m, a_{x2}^m, \ldots)$ [illegible] $a = (a_1, a_2, \ldots)$ [illegible] $|g_x a|$, [illegible]

$$g_x a = \sum_{j=1}^{\infty} \gamma$$

[illegible] $\hat{\psi}_j$, [illegible]

$$\hat{\gamma}_{xj} = \frac{\sum\sum^{(j)}_{i_1,i_2} Y_{i_1 i_2} K(i_1, i_2, j|x)}{\sum\sum^{(j)}_{i_1,i_2} \hat{\xi}_{i_1 i_2 j} K(i_1, i_2, j|x)}. \tag{7}$$

[illegible], $\sum\sum^{(j)}_{i_1,i_2}$ [illegible] (i_1, i_2) [illegible] $\hat{\xi}_{i_1 i_2 j} > 0$,

$$K(i_1, i_2, j|x) = K\left(\frac{\|x - X_{i_1}\|}{h_1}\right) K\left(\frac{\|x - X_{i_2}\|}{h_1}\right) K\left(\frac{Q_{i_1 i_2 j}}{h_2}\right), \tag{8}$$

K [illegible] h_1 [illegible] h_2 [illegible] (8), [illegible] (i_1, i_2), [illegible] $X_{i_1} - X_{i_2}$ [illegible] $\hat{\psi}_j$, [illegible] i_1 [illegible] i_2, [illegible] X_{i_1} [illegible] X_{i_2} [illegible] x. [illegible] $\hat{\gamma}_{xj}$ [illegible], h_1 [illegible] h_2.

3. [illegible]

3.1. *Consistency and convergence rates of estimators of g.* [illegible] g [illegible] x ([illegible] y [illegible] δ, [illegible]).

A[illegible] 2.

$$\sup_{y:\|y\|\le 1} |g(x + \delta y) - g(x)| \to 0 \quad \text{[illegible]} \quad \delta \downarrow 0, \tag{9}$$

[illegible] h [illegible], [illegible] c [illegible] A[illegible] 1,

$$h = h(n) \to 0 \quad \text{[illegible]} \quad nP(\|X - x\| \le c_1 h) \to \infty \quad \text{[illegible]} \quad n \to \infty,$$
$$\text{[illegible]} \ c_1 = c \ \text{[illegible]} \ K(c) > 0, \ \text{[illegible]} \ c_1 \in (0, c). \tag{10}$$

[illegible] $C > 0$, $x \in L_2(\mathcal{I})$ [illegible] $\alpha \in (0, 1]$, [illegible] $\mathcal{G}(C, x, \alpha)$ [illegible] g [illegible] $|g(x + \delta y) - g(x)| \le C\delta^\alpha$, [illegible] $y \in L_2(\mathcal{I})$ [illegible] $\|y\| \le 1$, [illegible] $0 \le \delta \le 1$. [illegible] (9) [illegible] g [illegible] $\mathcal{G}(C, x, \alpha)$.

[illegible] $\mathcal{X} = \{X_1, \ldots, X_n\}$ [illegible].

T[illegible] 1. *If Assumptions 1 and 2 hold, then $\hat{g}(x) \to g(x)$ in mean square, conditional on $\mathcal{X}$, and*

$$\sup_{g \in \mathcal{G}(C,x,\alpha)} E[\{\hat{g}(x) - g(x)\}^2 | \mathcal{X}] = o_p(1). \tag{11}$$

Furthermore, for all $\eta > 0$,

$$\sup_{g \in \mathcal{G}(C,x,\alpha)} P\{|\hat{g}(x) - g(x)| > \eta\} \to 0.$$

Moreover, if h *is chosen to decrease to zero in such a manner that*

$$h^{2\alpha} P(\|X - x\| \le c_1 h) \asymp n^{-1} \tag{12}$$

as $n \to \infty$, *then, for each* $C > 0$, *the rate of convergence of* $\hat{g}(x)$ *to* $g(x)$ *equals* $O_p(h^{2\alpha})$, *uniformly in* $g \in \mathcal{G}(C, x, \alpha)$:

$$\sup_{g \in \mathcal{G}(C,x,\alpha)} E[\{\hat{g}(x) - g(x)\}^2 | \mathcal{X}] = O_p(h^{2\alpha}), \tag{13}$$

$$\lim_{C_1 \to \infty} \limsup_{n \to \infty} \sup_{g \in \mathcal{G}(C,x,\alpha)} P\{|\hat{g}(x) - g(x)| > C_1 h^{\alpha}\} = 0. \tag{14}$$

[illegible] (11) [illegible] (13), [illegible] (X_i, ε_i), [illegible] $1 \le i < \infty$, [illegible] $Y_i = Y_i(g) = g(X_i) + \varepsilon_i$. [illegible] $E_g(\cdot|\mathcal{X})$ [illegible] $(X_i, Y_i(g))$, [illegible] $\mathcal{X}$. [illegible] 5.1 [illegible] (12), [illegible] g [illegible] (2007) [illegible] (14) [illegible]

[illegible] 2. *If the error* ε *in* (1) *is normally distributed, and if, for a constant* $c_1 > 0$, $nP(\|X - x\| \le c_1 h) \to \infty$ *and* (12) *holds, then, for any estimator* $\tilde{g}(x)$ *of* $g(x)$, *and for* $C > 0$ *sufficiently large in the definition of* $\mathcal{G}(C, x, \alpha)$, *there exists a constant* $C_1 > 0$, *such that*

$$\liminf_{n \to \infty} \sup_{g \in \mathcal{G}(C,x,\alpha)} P\{|\tilde{g}(x) - g(x)| > C_1 h^{\alpha}\} > 0.$$

[illegible] $\mathcal{G}(C, x, \alpha)$. [illegible] x [illegible] $P(\|X - x\| \le u)$ [illegible] (12) [illegible] 5.1.

3.2. *Consistency of derivative estimator.* [illegible] $\hat{\gamma}_{xj}$. [illegible]

$$q_{12j} = 1 - \frac{|\square (X_1 - X_2)\psi_j|^2}{\|X_1 - X_2\|^2}$$

[illegible] Q_{12j} [illegible] $\hat{\xi}_{i_1 i_2}$ [illegible] ξ_j [illegible] $\hat{\xi}_{i_1 i_2}$ [illegible] $k_{i_1 i_2 j}$ [illegible] $K(i_1, i_2, j|x)$, [illegible] (8), [illegible] $Q_{i_1 i_2 j}$ [illegible] $q_{i_1 i_2 j}$.

A 3.

( ) $_{t\in\mathcal{I}} E\{X(t)^4\} < \infty$;

( ) m $\theta_1, \ldots, \theta_{j+1}$;

( ) $|g(x+y) - g(x) - g_x y| = o(\|y\|)$ $\|y\| \to 0$;

( ) r $\xi_{1j} - \xi_{2j}$ r r ;

( ) K $[0,1]$, r $0 < K(0) < \infty$;

( ) $h_1, h_2 \to 0$ n $n^{1/2}$ m $(h_1, h_2) \to \infty$ $(nh_1)^2 E(k_{i_1 i_2 j}) \to \infty$.

r X r r r V, ψ_j m r $\hat{\psi}_j$ 3.1, r m m, A m 4( ) $\|\hat{\psi}_j - \psi_j\|$ r r r -n r . m , m , r r m m r .

X r r m- $X\psi_j$ (r r r m X) θ_j r r r (1) , A m 3( ) m $n^{-\varepsilon} = O(h_j)$ r $j = 1, 2$ r $\varepsilon > 0$ r r (2) . , r r r - m n^{-1}. , (1) $P(\|x - X\| \le h_1) = O(h_1^{C_1})$ r $C_1 > 0$. A , 3( ) m $nh_1 P(\|x - X\| \le C_2 h_1) \to \infty$ r m $C_2 > 0$, , r (1), $nh^{C_1+1} \to \infty$ r $C_1 > 0$. (2) r h_1. r r (1) m $P(q_{12j} \le h_2) = O(h_2^{C_1})$ r $C_1 > 0$, 3( ) m $nP(q_{12j} \le C_2 h_2) \to \infty$ r m $C_2 > 0$, , (1), m r .

3. *If Assumption 3 holds, then $\hat{\gamma}_{xj} \to \gamma_{xj}$ in probability.*

(5), $e = \sum_{j=1}^{j_0} e_j \psi_j$ $\sum_j e_j^2 = 1$ $j_0 < \infty$, - r e x $g_x e = \sum_j e_j \gamma_{xj}$; (6), e $a_j = e_j, 1 \le j \le j_0, a_j = 0, j > j_0$. A m- 3 r $j \le j_0$, m m 3 m r $\hat{g}_x e = \sum_j e_j \hat{\gamma}_{xj}$ x e (., $\hat{g}_x e \to g_x e$ r). A r m r - r e, r r $\{\psi_1, \ldots, \psi_{j_0}\}$ m .

r r $\hat{g}_x$, $\hat{g}_x a = \sum_{j \le r} \hat{\gamma}_{xj} a_j$ ($r \ge 1$ r $a = \sum_j a_j \psi_j$), m r r m g_x,

$g_x a = \sum_j \gamma_{xj} a_j$, ... 3 ... j, ... $\sum_j \gamma_{xj}^2 < \infty$, ... (...) ... $r = r(n, x)$ ... : $r(n, x) \to \infty$ ... $n \to \infty$; ... $\|a\| < \infty$, $\hat{g}_x a - g_x a \to 0$ ... ; ... $\hat{g}_x \to g_x$ ... $n \to \infty$, ... $r(n, x)$, ... j ... 3, ...

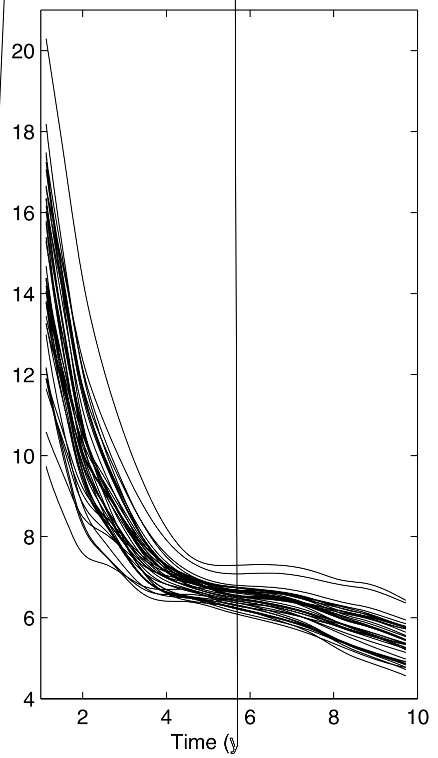
4. A ... cat ... f f ... ct ... a de ... at e e t ... at ... t g ... t data.

... (1958), ... (1984), ... (1992), ... (1998) ... (2005) ...

... 18 (...), ... 10 (...), ... 39 ... : ... ? ... ? ...

... 39 ... (1954), ... 10 ... 15 ... 10 ... {1, 1.25, 1.5, 1.75, 2, 3, 4, 5, 6, 7, 8, 8.5, 9, 9.5, 10}, ... $\{s_j\}_{j=1,\ldots,15}$. ... $X_{ij} = (h_{i,j+1} - h_{ij})/(t_{j+1} - t_j)$, ... h_{ij} ... s_j ... i ..., $t_j = (s_j + s_{j+1})/2$, $i = 1, \ldots, 39$, $j = 1, \ldots, 14$. ... (2003) ... (2005).

A ... B ...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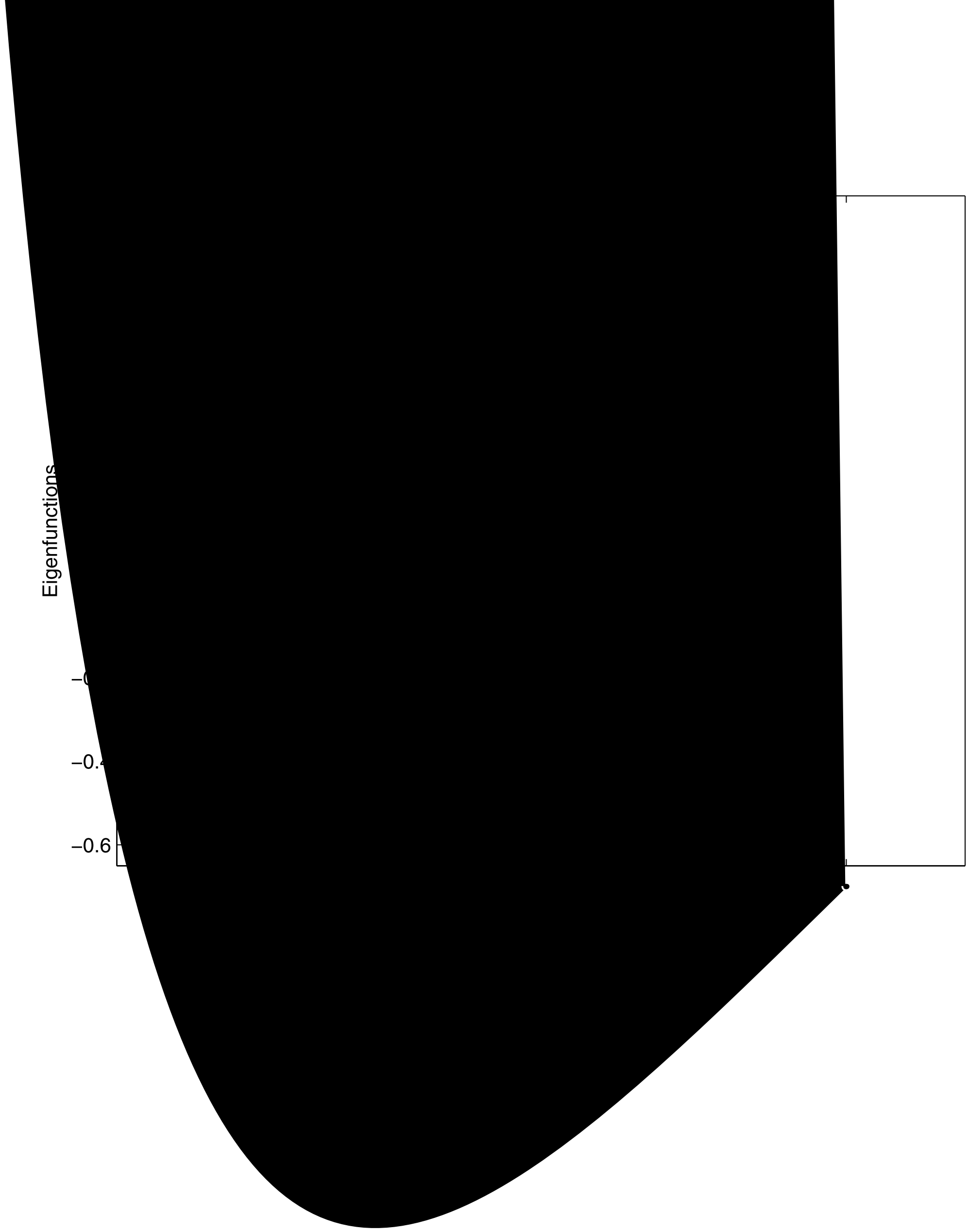
Time (y



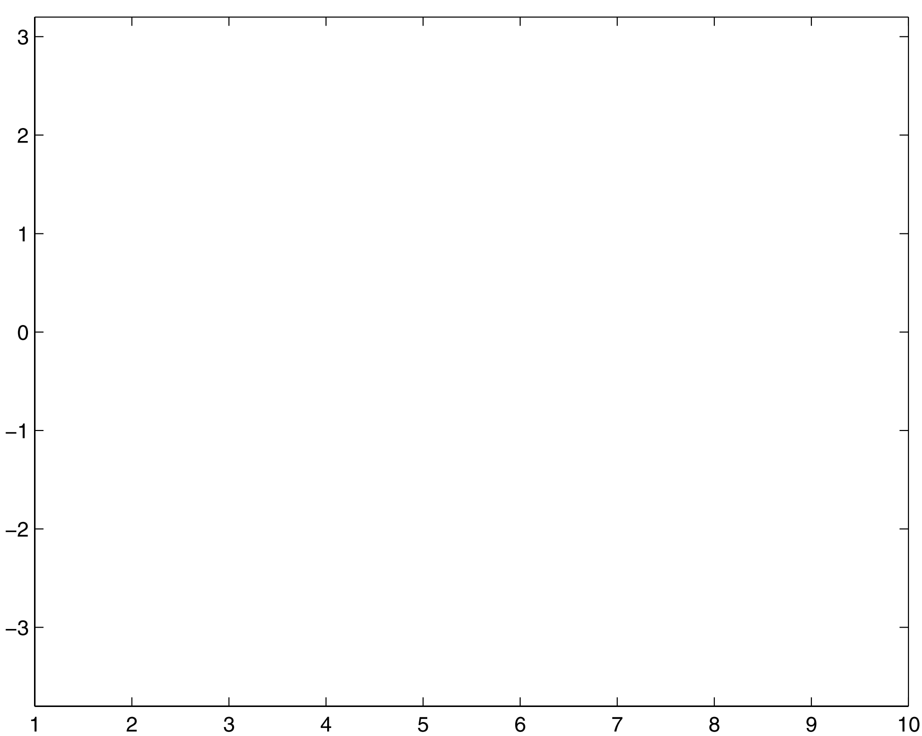
$K j$ ψ ($i,$

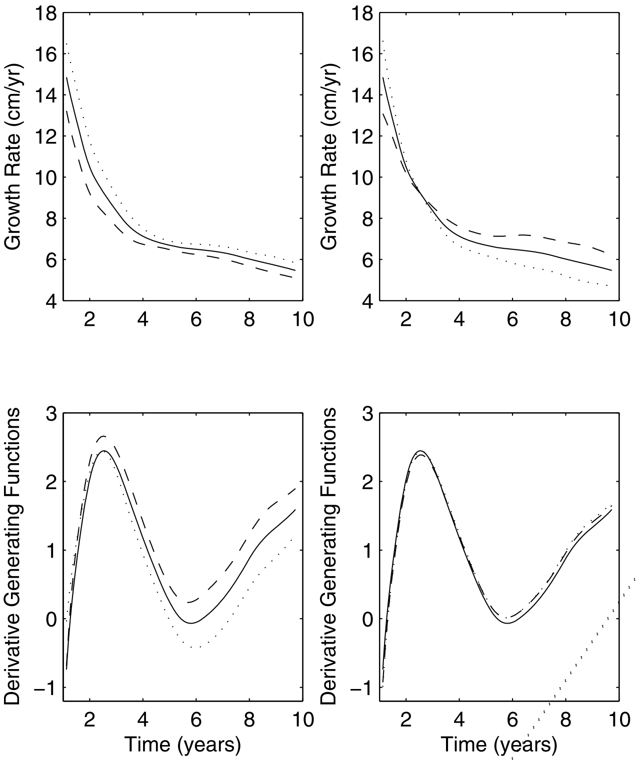
Smooth estimates of the first three eigenfunctions for the velocity growth curves [illegible] $\int g_i^*(t)z(t)\,dt = \sum_{j=1}^{K} \gamma_{X_i,j} z_j$, [illegible] X_i [illegible] z [illegible] z [illegible] g_i^*. [illegible] g_i^* [illegible] *derivative generating function* [illegible] X_i. [illegible] $K = 3$ [illegible] X_i [illegible] z. [illegible]

$$\hat{g}_i^*(t) = \sum_{j=1}^{K} \hat{\gamma}_{X_i,j} \hat{\psi}_j(t) \tag{15}$$

[illegible] $\gamma_{X_i,j}$ [illegible] $\psi_j(t)$ [illegible] (3) [illegible] (7).

[illegible] $\hat{g}_i^*$ [illegible] $K = 3$ [illegible] 3 [illegible] 39 [illegible] X_i [illegible]





Growth Rate (cm/yr)
Growth Rate (cm/yr)
Derivative Generating Functions
Derivative Generating Functions
Time (years)
Time (years)

[illegible]

5. Add t a e t a d f.

5.1. *Bounds on* $P(\|X-x\|\le u)$. [illegible] $P(\|X-x\|\le u)$ [illegible] $u\to 0$ [illegible]; [illegible] u. [illegible] (19) [illegible] $\hat{g}(x)$ [illegible] $g(x)$ [illegible] X [illegible] (14) [illegible] n.

[illegible] X [illegible] X [illegible] $X = \sum_j \xi_j\psi_j$, [illegible] $(\xi_j)=\theta_j$ [illegible] θ_j, $j\ge 1$, [illegible] j [illegible] ξ_j [illegible].

A [illegible] 4. [illegible] $B,\beta>0$,

$$\theta_j = -Bj^{\beta}+o(j^{\beta}) \qquad j\to\infty, \tag{16}$$

[illegible] $\eta_j=\xi_j/\theta_j^{1/2}$ [illegible] η, [illegible]

$$B_1u^b \le P(|\eta|\le u)\le B_2u^b \quad [\text{illegible}]\ u>0, \qquad P(|\eta|>u)\le B_3(1+u)^{-B_4} \quad [\text{illegible}]\ u>0, \quad [\text{illegible}]\ B_1,\dots,B_4,b>0. \tag{17}$$

[illegible] $x=0$, [illegible] b, B [illegible] β [illegible] (16) [illegible] (17), [illegible]

$$\pi(u) = [\text{illegible}]\left\{-\frac{b\beta}{\beta+1}\left(\frac{2}{B}\right)^{1/\beta} |[\text{illegible}]\, u|^{(\beta+1)/\beta}\right\}. \tag{18}$$

[illegible] 4. *If* (16) *and* (17) *hold, then, with* $\pi(u)$ *given by* (18),

$$P(\|X\|\le u) = \pi(u)^{1+o(1)} \qquad \text{as } u\downarrow 0. \tag{19}$$

[illegible] 1 [illegible] 3, [illegible] θ_j [illegible] (16), [illegible] ξ_j [illegible] (17), [illegible] h [illegible] (12) [illegible]

$\hat{g}(x)$ $g(x)$

$$h^{2\alpha} = (-2\alpha|\ h|)$$
$$= -\{1+o(1)\}2\alpha \left(\frac{\beta+1}{b\beta}\right)^{\beta/(\beta+1)} \left(\frac{B}{2}\right)^{1/(\beta+1)} (\ n)^{\beta/(\beta+1)} .$$

β, n^{-1}, β (16) (17) $\theta_j = (-Bj^{\beta})$, η 4 $\{\phi_j\}$ $\beta = b = 1$ $\pi(u) = \{-c(\ u)^{(\beta+1)/\beta}\} = u^{-c(\ u)^{1/\beta}}$, $c > 0$, 0. η X.

4 $x = 0$, $P(\|X - x\| \le u)$ u, (19) $x = 0$ X $X_1 - X_2$, (19) (19), $P(\|X_1 - x\| \le u)$ x X_2. (19) $x = 0$. x, $x = \sum_j \theta_j^{1/2} x_j$ x_j' x_j'

θ_j (16), $P(\|X - x\| \le u)$ $(-C_1 u^{-C_2})$ u 0, $(-C_1|\ u|^{C_2})$ 3 $C_1, C_2 > 0$. $x = 0$, (2003), (1952). u^{C_3} $(-C_1 u^{-C_2})$ $C_3 > 0$ (2007),

5.2. *Proof of Theorem* 1. σ^2 ε (1). $N_j = \sum_i K_i(x)^j$ $j = 1, 2,$ $N_2 \le K(0) N_1$, $K(\cdot)$

... $[0, c]$. ...,

$$
\begin{aligned}
&E[\{\hat{g}(x) - g(x)\}^2 | \mathcal{X}] \\
&\quad = [E\{\hat{g}(x) | \mathcal{X}\} - g(x)]^2 + \operatorname{var}(\hat{g}(x) | \mathcal{X}) \\
&\quad \le \max_{i=1,\dots,n} |g(X_i) - g(x)| I(\|X_i - x\| \le ch) + \frac{\sigma^2 \sum_i K_i^2(x)}{\{\sum_i K_i(x)\}^2} \\
&\quad \le \sup_{y:\|y\| \le ch} |g(x) - g(x+y)|^2 + \frac{\sigma^2 K(0)}{N_1}.
\end{aligned}
\tag{20}
$$

... g ... x ..., (9) ... (20) ... $K_i(x) \ge K_i(x) I(\|X_i - x\| \le c_1 h) \ge K(c_1) I(\|X_i - x\| \le c_1 h)$, ... c_1 ... (A2). ..., (10) ... $N_1^{-1} \to 0$ ... 1, ... $E(N_i^{-1}) \to 0$. ... (20), ... (13) ... (14), ... (20)

$$
\sup_{g \in \mathcal{G}(C,x,\alpha)} E[\{\hat{g}(x) - g(x)\}^2 | \mathcal{X}] \le C^2 (ch)^{2\alpha} + \frac{\sigma^2 K(0)}{N_1}
$$

$$
\le C^2 (ch)^{2\alpha} + \frac{\sigma^2 K(0)\{1 + o_p(1)\}}{K(c_1) n P(\|X - x\| \le c_1 h)}
$$

... $E(N_1^{-1}) \le E[\{\sum_i I(\|X_i - x\| \le c_1 h)\}^{-1}] \asymp \{nP(\|X - x\| \le c_1 h)\}^{-1}$.

5.3. *Proof of Theorem* 2. ... $x = 0$. ... f ... B_1, ... $[-B_2, B_2]$, ... f ... B_3 ... $g_1 \equiv 0$ ... $g_2(y) = h^\alpha f(\|y\|/h)$. ... $\|y\| \le h$ ... $0 < \alpha \le 1$,

$$
\begin{aligned}
|g_2(y) - g_2(0)| &= h^\alpha |f(\|y\|/h) - f(0)| \le h^\alpha B_1 \|y\|/h \le h^\alpha B_1 (\|y\|/h)^\alpha \\
&= B_1 \|y\|^\alpha,
\end{aligned}
$$

... $\|y\| > h$,

$$
|g_2(y) - g_2(0)| \le 2h^\alpha B_3 \le 2B_3 \|y\|^\alpha.
$$

..., $g_2 \in \mathcal{G}(C, 0, \alpha)$ ... $\max(B_1, 2B_3) \le C$.

...

(21) $$P(\rho > 1) \quad 1 \quad n \to \infty.$$

$$2 \quad \rho = \sum_{i=1}^{n} \{g_2(X_i)^2 - 2\varepsilon_i g_2(X_i)\},$$

$\mathcal{X}$, $s_n^2 = \sum_i g_2(X_i)^2$ $4s_n^2$. (21)

(22) $$\lim_{B\to\infty} \lim_{n\to\infty} P(s_n^2 > B) = 0$$

2 (22).

B_2 f $0 < B \le c_1$, $|g_2(x)| \le B_3 h^{\alpha} I(\|x\| \le c_1 h)$,

(23) $$s_n^2 \le B_3^2 h^{2\alpha} \sum_{i=1}^{n} I(\|X_i\| \le c_1 h).$$

$nP(\|X\| \le c_1 h) \to \infty$,

$$\frac{\sum_i I(\|X_i\| \le c_1 h)}{nP(\|X\| \le c_1 h)} \to 1$$

(12) (23) (22).

5.4. *Proof of Theorem* 3. $K_{i_1 i_2 j}$ $K(i_1, i_2, j|x)$. A 3( )

(24) $$K_{i_1 i_2 j} = 0, \quad : \|X_{i_1} - x\| \le h_1, \quad \|X_{i_2} - x\| \le h_1 \quad Q_{i_1 i_2} \le h_2.$$

$\delta > 0$, $s(\delta)$ $|g(x+y) - g(x) - g_x y|$ y $\|y\| \le \delta$. A 3( ),

(25) $$\delta^{-1} s(\delta) \to 0 \quad \delta \downarrow 0.$$

$\mathcal{E}_{i_1 i_2}$ $\|X_{i_k} - x\| \le h_1$ $k = 1, 2$. $\mathcal{E}_{i_1 i_2}$,

$$|g(X_{i_1}) - g(X_{i_2}) - g_x(X_{i_1} - X_{i_2})| \le 2s(h_1).$$

$\varepsilon_{i_1 i_2} = \varepsilon_{i_1} - \varepsilon_{i_2}$ $\mathcal{E}_{i_1 i_2}$,

$$|Y_{i_1} - Y_{i_2} - \{g_x(X_{i_1} - X_{i_2}) + \varepsilon_{i_1 i_2}\}| \le 2s(h_1).$$

[illegible] $\xi_{i_1i_2j}=\xi_{i_1j}-\xi_{i_2j}$, [illegible] $g_x(X_{i_1}-X_{i_2})=\sum_k \xi_{i_1i_2k}\gamma_{xk}$, [illegible] (24), [illegible],

$$\begin{aligned}&\Bigg|\sum\sum^{(j)}_{i_1,i_2}(Y_{i_1}-Y_{i_2})K_{i_1i_2j}\\&\quad-\Bigg\{\sum\sum^{(j)}_{i_1,i_2}K_{i_1i_2j}\sum_{k=1}^{\infty}\xi_{i_1i_2k}\gamma_{xk}+\sum\sum^{(j)}_{i_1,i_2}\varepsilon_{i_1i_2}K_{i_1i_2j}\Bigg\}\Bigg|\\&\quad\le 2s(h_1)\sum\sum^{(j)}_{i_1,i_2}K_{i_1i_2j}.\end{aligned}\tag{26}$$

[illegible],

$$\begin{aligned}|\hat{\xi}_{i_1i_2j}-\xi_{i_1i_2j}|&=\Big|\int(X_{i_1}-X_{i_2})(\hat{\psi}_j-\psi_j)\Big|\\&\le\|X_{i_1}-X_{i_2}\|\|\hat{\psi}_j-\psi_j\|\le 2h_1\|\hat{\psi}_j-\psi_j\|,\end{aligned}\tag{27}$$

[illegible] $\varepsilon_{i_1i_2}$ [illegible] (24), (26) [illegible] (27), [illegible]

$$\begin{aligned}&\Bigg|\sum\sum^{(j)}_{i_1,i_2}(Y_{i_1}-Y_{i_2})K_{i_1i_2j}\\&\quad-\Bigg\{\gamma_{xj}\sum\sum^{(j)}_{i_1,i_2}\hat{\xi}_{i_1i_2j}K_{i_1i_2j}\\&\quad+\sum\sum^{(j)}_{i_1,i_2}K_{i_1i_2j}\sum_{k:k\neq j}\xi_{i_1i_2k}\gamma_{xk}+\sum\sum^{(j)}_{i_1,i_2}\varepsilon_{i_1i_2}K_{i_1i_2j}\Bigg\}\Bigg|\\&\quad\le 2\{s(h_1)+|\gamma_{xj}|h_1\|\hat{\psi}_j-\psi_j\|\}\sum\sum^{(j)}_{i_1,i_2}K_{i_1i_2j}.\end{aligned}\tag{28}$$

[illegible],

$$\begin{aligned}&\Bigg|\sum\sum^{(j)}_{i_1,i_2}K_{i_1i_2j}\sum_{k:k\neq j}\xi_{i_1i_2k}\gamma_{xk}\Bigg|\\&\quad=\Bigg|\sum\sum^{(j)}_{i_1,i_2}K_{i_1i_2j}\sum_{k:k\neq j}\gamma_{xk}\int(X_{i_1}-X_{i_2})\psi_k\Bigg| \quad [illegible]\\&\quad\le\sum\sum^{(j)}_{i_1,i_2}K_{i_1i}\end{aligned}$$

$$\leq \|g_x\| \sum_{i_1,i_2} \square^{(j)} K$$

[illegible] (32) [illegible] $\hat{\gamma}_{xj}$ [illegible] (7). [illegible] (27) [illegible]

$$\sum_{i_1,i_2}^{(j)} \hat{\xi}_{i_1 i_2 j} K_{i_1 i_2 j} \geq \sum_{i_1,i_2}^{(j)} \max(0, \xi_{i_1 j} - \xi_{i_2 j} - 2h_1 \|\hat{\psi}_j - \psi_j\|) K_{i_1 i_2 j}$$

$$\geq \sum_{i_1,i_2}^{(j)} \max(0, \xi_{i_1 j} - \xi_{i_2 j}) K_{i_1 i_2 j} \tag{33}$$

$$- 2h_1 \|\hat{\psi}_j - \psi_j\| \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j}.$$

[illegible] $\sum\sum_{i_1,i_2}^{(j)}$ [illegible] (i_1, i_2) [illegible] $\hat{\xi}_{i_1 i_2 j} > 0$. [illegible] 4( ), ( ) [illegible] ( ), [illegible] $B > 0$,

$$\sum_{i_1,i_2}^{(j)} \max(0, \xi_{i_1 j} - \xi_{i_2 j}) K_{i_1 i_2 j} \geq \{1 + o_p(1)\} B h_1 \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j}. \tag{34}$$

[illegible] 3( ), $n^{-1/2}/\min(h_1, h_2) \to 0$. [illegible] 3( ) [illegible] ( ), [illegible]

$$\|\hat{\psi}_j - \psi_j\| = O_p(n^{-1/2}). \tag{35}$$

[illegible] (33) (35) [illegible]

$$\sum_{i_1,i_2}^{(j)} \hat{\xi}_{i_1 i_2 j} K_{i_1 i_2 j} \geq \{1 + o_p(1)\} B h_1 \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j} \tag{36}$$

[illegible] B [illegible] (34). [illegible] (7). [illegible] (7), (25), (32) [illegible] (36), [illegible]

$$\hat{\gamma}_{xj} = \gamma_{xj} + O_p\left(\frac{\sum\sum_{i_1,i_2}^{(j)} \varepsilon_{i_1 i_2} K_{i_1 i_2 j}}{h_1 \sum\sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j}}\right) + o_p(1). \tag{37}$$

[illegible] (37), [illegible] X_i [illegible]

$$O_p\left[\left\{h_1^2 \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j}\right\}^{-1}\right] = O_p[\{(nh_1)^2 E(k_{i_1 i_2 j})\}^{-1}] = o_p(1),$$

[illegible] 3( ). [illegible] (37) [illegible] $\hat{\gamma}_{xj} = \gamma_{xj} + o_p(1)$, [illegible] 3.

5.5. *Proof of Theorem* 4. [illegible] $t \in (0,1)$ [illegible] $D_t = (\sum_j \theta_j^{1-t})^{-1}$,

$$(38)\qquad P(\|X\| \le u) = P\left(\sum_{j=1}^{\infty} \theta_j \eta_j^2 \le u^2\right) \begin{cases} \le \prod_{j=1}^{\infty} P(\theta_j \eta_j^2 \le u^2), \\ \ge \prod_{j=1}^{\infty} P(\theta_j^t \eta_j^2 \le D_t u^2), \end{cases}$$

[illegible]

$$P\left(\sum_{j=1}^{\infty} \theta_j \eta_j^2 \le u^2\right) = P\left(\sum_{j=1}^{\infty} \theta_j^{1-t}(\theta_j^t \eta_j^2 - D_t u^2) \le 0\right)$$
$$\ge P(\theta_j^t \eta_j^2 \le D_t u^2 \text{ [illegible] } j).$$

[illegible] $J = J(u)$ [illegible] $u/\theta_j^{1/2} \le \zeta$, [illegible] ζ [illegible] $B_1 u^b \le P(|\eta| \le u) \le B_2 u^b$ [illegible] $0 \le u \le \zeta$. [illegible],

$$(39)\qquad \begin{aligned} \prod_{j=1}^{\infty} P(\theta_j \eta_j^2 \le u^2) &\le \prod_{j=1}^{J} P(|\eta| \le u\theta_j^{-1/2}) \\ &= u^{bJ} \exp\left(\frac{1}{2} bB \sum_{j=1}^{J} j^{\beta} + o(J^{\beta+1})\right) \\ &= \exp\left(-\frac{bB\beta}{2(\beta+1)} J^{\beta+1} + o(J^{\beta+1})\right) \\ &= \pi(u)^{1+o(1)} \end{aligned}$$

[illegible] $u \downarrow 0$, [illegible] π [illegible] (18).

[illegible] J [illegible] $D_t^{1/2} u/\theta_j^{t/2} \le \zeta$. [illegible] (39), [illegible]

$$(40)\qquad \begin{aligned} &\prod_{j=1}^{J} P(\theta_j^t \eta_j^2 \le D_t u^2) \\ &= \exp\left(-\frac{b\beta}{\beta+1}\left(\frac{2}{Bt}\right)^{1/\beta} |\log u|^{(\beta+1)/\beta} + o\left(|\log u|^{(\beta+1)/\beta}\right)\right) \\ &= \pi(u)^{t^{-1/\beta}+o(1)}. \end{aligned}$$

A[illegible], [illegible] $j \ge J+1$,

$\pi o($

$B_5 = B_5(t) \in (0, 1)$, $\pi_j \in (0, B_5)$ $j \geq J + 1$,

$$1 - \pi_j = \exp\left(-\sum_{k=1}^{\infty} \frac{\pi_j^k}{k}\right) \geq \exp(-B_6 \pi_j)$$

$$\prod_{j=J+1}^{\infty}(1 - \pi_j) \geq \exp\left(-B_6 \sum_{j=J+1}^{\infty} \pi_j\right) \geq \exp\left(-B_7 \sum_{j=J+1}^{\infty} (\theta_j^{t/2}/u)^{B_4}\right),$$

(40). (40), $t \in (0, 1)$ (40), 1, $u \downarrow 0$,

$$P(\theta_j^t \eta_j^2 \leq D_t u^2) = \pi(u)^{1+o(1)}. \tag{42}$$

(38), (39) (42) (19).

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