

# RESEARCH STATEMENT: GEOMETRIC RAMIFICATION THEORY

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My current research focuses on algebraic geometry, arithmetic geometry and number theory, especially on geometric ramification theory. In Section 1, I will recall the current developments in geometric ramification theory and my research results. In Section 2, I will propose my future research plan. In Section 3, I will discuss the possible academic methods which may be used in the future research project.

## 1. CURRENT STATE OF RESEARCH

1.1. **Milnor formula.** In algebraic geometry, the Milnor formula computes the *total dimension* of the vanishing cycle of a constructible étale sheaf  $F$  on a regular flat scheme of finite type  $X$  over a trait  $S$  with perfect residue field. Here, the total dimension means the sum of the dimension and the Swan conductor. Problem of this type was considered previously in two cases:

- (1)  $X$  may not be smooth over  $S$ , and  $F$  is locally constant. The formula is then known as the classical Milnor formula.
- (2)  $X$  is smooth over  $S$ , but  $F$  may have ramification. The relative curve case is due to P. Deligne and G. Laumon [Lau81], A. Abbes [Ab00a], K. Kato and T. Saito [TS15].

In [Y14a, Y14b], we considered at the same time a non-smooth scheme  $X$  over  $S$  and a non-locally constant sheaf  $F$ . We proposed a logarithmic version of the Milnor formula and proved this formula in the geometric case. Let  $X$  be a regular scheme and  $D$  a simple normal crossing divisor on  $X$ . Let  $S$  be a regular scheme purely of dimension 1 and  $s$  a closed point of  $S$  with perfect residue field of positive characteristic  $p > 0$ . Assume that  $f : X \rightarrow S$  is a flat morphism of finite type which has an isolated log-singular point at  $x_0 \in X_s$ , i.e., there exists an open neighborhood  $V$  of  $x_0$  in  $X$  such that  $f|_{V \setminus \{x_0\}} : V \setminus \{x_0\} \rightarrow S$  is smooth and that  $D \cap V \setminus \{x_0\}$  is a divisor on  $V \setminus \{x_0\}$  with simple normal crossings relatively to  $S$ . Let  $j : U = X \setminus D \rightarrow X$  the open immersion and  $F$  a locally constant and constructible sheaf of  $\mathbb{F}$ -vector spaces on  $U$  where  $\mathbb{F} \neq \mathbb{F}_p$  is a prime number. Assume that  $F$  is tamely ramified along the boundary  $D$ . Then it is conjectured in [Y14a, Conjecture 1.12, P12] that

$$(-1)^n \dim_{\text{tot}} \mathbf{R}_{x_0}(j_! F) = \text{rank } F \cdot \text{length}_{\mathcal{O}_{X,x}}(\text{Ext}_{\mathcal{O}_X}^1(\frac{1}{X/S}(\log D), \mathcal{O}_X))_{x_0}, \quad (1.1.1)$$

where  $\dim_{\text{tot}} = \dim + \text{Sw}$  is the sum of dimension and the Swan conductor,  $\mathbf{R}_{x_0}$  is the vanishing cycle functor for  $f$ . We call (1.1.1) the logarithmic Milnor formula. When  $F$  is the constant étale sheaf  $\mathbb{F}$  on  $U$ , we proved that the above conjecture is equivalent to the following classical conjectural Milnor formula (cf. [Del73, Conjecture 1.9, P200])

$$(-1)^n \dim_{\text{tot}} \mathbf{R}_{x_0}(\mathbb{F}) = \text{length}_{\mathcal{O}_{X,x}}(\text{Ext}_{\mathcal{O}_X}^1(\frac{1}{X/S}, \mathcal{O}_X))_{x_0}. \quad (1.1.2)$$

In [Del73], P. Deligne proved the formula (1.1.2) in the following cases:

- (a)  $n = 0$ , or
- (b)  $X/S$  has an ordinary quadratic singularity at  $x$ , or
- (c)  $S$  is of equal characteristic.

In [Org03, Théorème 0.8, P1741], F. Orgogozo showed that the conductor formula of Bloch implies the Milnor formula. In [KS04, Corollary 6.2.7, P142], K. Kato and T. Saito showed that the conductor formula is a consequence of an embedded resolution of singularities in a strong sense for

the reduced closed fiber. Hence, the Milnor formula is true if we assume an embedded resolution. In particular, the Milnor formula is true if the relative dimension is equal to one or two.

When the sheaf  $F$  is not tame, the situation is much more complicated. A general program, initiated by P. Deligne, is to try to define the *characteristic cycle*  $\text{Char}(j_!F)$  as a cycle of codimension  $\dim(X)$  in the cotangent bundle  $T^*X$  such that

- (a) When  $X$  is proper over a perfect field  $k$ ,

$$c(U, F) = (\text{Char}(j_!F), T_X^*X)_{T^*X}. \quad (1.1.3)$$

- (b) Let  $f: X \rightarrow C$  be a flat morphism from a smooth scheme  $X$  to a smooth curve  $C$  over a perfect field  $k$  of characteristic  $p > 0$ . Let  $F$  be a locally constant and constructible sheaf of  $\mathbb{F}$ -vector spaces on  $U$ . Assume that  $x_0$  is an isolated characteristic point of  $f$  with respect to  $j_!F$  (cf. [TS15, Definition 3.7]). Then

$$-\dim_{\text{tot}} \mathbf{R} \pi_{x_0}(j_!F) = (\text{Char}(j_!F), df(X))_{T^*X, x_0}. \quad (1.1.4)$$

When  $F$  is tame, (a) is well-known and (b) follows from the logarithmic Milnor formula. When  $X$  is a smooth curve, (a) is the so-called Grothendieck–Ogg–Shararevich formula. When  $X$  is a smooth surface, P. Deligne and G. Laumon defined the characteristic cycle implicitly in [Lau83] under the “non-fierce” assumption.

Recently, similar to the theory on manifolds [Kas90], A. Beilinson defined the *singular support*  $SS(F)$  for any étale constructible sheaf  $F$  on a smooth algebraic variety  $X$  over an arbitrary base field  $k$  in [Bei15]. The singular support  $SS(F)$  is the smallest closed conical subset of the cotangent bundle  $T^*X$  such that locally on  $X$ , every function  $f: X \rightarrow \mathbb{A}_k^1$  with  $df$  disjoint from  $SS(F)$  is locally acyclic relative to  $F$ . It is proved in [Bei15] that  $SS(F)$  is of dimension  $n = \dim(X)$  if  $X$  is connected. In the proof of the existence of singular support [Bei15], A. Beilinson used Brylinski’s Radon transform. We point out that there should be a similar Radon transform theorem for Grassmann varieties. We refer to [GR04] for the case of Grassmann manifold. Based on Beilinson’s work [Bei15], T. Saito [TS15, TS16a, TS16b, TS16c] constructed an  $n$ -cycle  $CC(F)$  with  $\mathbb{Z}$ -coefficients in  $T^*X$  supported on the singular support of  $F$ , which is characterized by a *Milnor formula* [TS16c, Theorem 4.9]. Usually  $CC(F)$  is called the *characteristic cycle* of  $F$ . If  $X$  is projective, then the intersection of  $CC(F)$  with the zero-section of  $T^*X$  computes the Euler–Poincaré characteristic of  $F$  [TS16c, Theorem 6.13].

It’s very natural to ask the following questions:

**Problem 1.1.** Define singular supports and characteristic cycles for constructible étale sheaves on a scheme which is smooth over a trait.

**Problem 1.2.** Try to construct an equivariant version of characteristic cycle or consider similar problems on Deligne–Mumford stacks. If this can be done, then we can define characteristic classes on varieties with quotient singularities. Another approach for defining characteristic classes on singular varieties is to use motivic cohomology.

**1.2. Semi-continuity.** In joint work with H. Hu [HY16], we extend the definition of singular support to a relative case. As an application, we proved the generic constancy for singular support and for characteristic cycle of a constructible étale sheaf on a smooth fibration. But their lower semi-continuity property fails, which is different from Deligne and Laumon’s semi-continuity theorem for Swan conductor [Lau83]. A direct generalization of Deligne and Laumon’s result is obtained in another joint work with H. Hu [HY15]. Here is a sketch:

Let  $X$  be a smooth scheme over an algebraically closed field  $k$  of characteristic  $p > 0$  and  $U = X - D$  be the open complement of a reduced divisor  $D$  in  $X$ . For a smooth sheaf  $F$  of  $\mathbb{F}$ -modules ( $\neq q$ ), we define the total dimension divisor  $DT_X(j_!F)$ , which roughly controls the ramification information of  $F$  at generic points of irreducible components of  $D$ . In [HY15], we considered two problems:

(Q1) Compare the pull-back of the total dimension divisor of an étale sheaf and the total dimension divisor of the pull-back of the sheaf,

(Q2) Study the semi-continuity of total dimension divisors of  $\ell$ -adic sheaves in higher relative dimension.

For problem (Q1), in his paper [TS13], T. Saito proved a pull-back formula. It says that if  $i : C \hookrightarrow X$  is a closed immersion from a smooth curve to a smooth scheme over  $k$  and if  $i$  is  $SS(j_!F)$ -transversal and  $C$  meets  $D$  properly, then one has a pull-back formula for  $DT$ , i.e.,  $i^*DT(j_!F) = DT(i^*j_!F)$ . In general, if we have a morphism  $f : Y \rightarrow X$  from a smooth scheme to  $X$ , then we get the inequality  $f^*DT(j_!F) \geq DT_Y(f^*j_!F)$ . For the proof of this result, we combined the methods of cut-by-curve and local fibration, and we reduced the problem to the case of relative dimension 1. Then the inequality follows from Deligne and Laumon's semi-continuity property for Swan conductors.

For the second problem (Q2), the relative 1 dimension case is due to P. Deligne and G. Laumon [Lau81]. In our work, we essentially use Deligne and Laumon's result and T. Saito's pull-back formula.

**1.3. Epsilon factor.** The L-function and the epsilon factor are two important objects in geometric Langlands program or Galois representation theory. Ramification theory will appear behind these objects in a very natural way.

Let  $k$  be a finite field of characteristic  $p$  and let  $f : X \rightarrow \text{Spec}(k)$  be a smooth projective morphism purely of dimension  $d$ . Let  $\bar{k}$  be a finite field of characteristic  $\neq p$ . For a constructible complex  $F$  of  $\ell$ -modules on  $X$ , let  $D(F)$  be the dual  $R\text{Hom}(F, K_X)$  of  $F$  where  $K_X = Rf^! \mathbb{Z}$  is the dualizing complex. The L-function  $L(X, F, t)$  satisfies the following functional equation

$$L(X, F, t) = (X, F) \cdot t^{-\langle X_{\bar{k}}, F \rangle} \cdot L(X, D(F), t^{-1}), \quad (1.3.1)$$

where

$$(X, F) = \det(-\text{Frob}_k; R(X_{\bar{k}}, F))^{-1} \quad (1.3.2)$$

is the epsilon factor (the constant term in the functional equation (1.3.1)) and  $\langle X_{\bar{k}}, F \rangle$  is the Euler characteristic of  $F$ . Let  $cc_X(F) \in CH_0(X)$  be the characteristic class of  $F$  defined in [TS16a, Definition 5.7]. In the functional equation (1.3.1), both  $\langle X_{\bar{k}}, F \rangle$  and  $(X, F)$  are related to the ramification theory. Indeed,  $\langle X_{\bar{k}}, F \rangle = \deg cc_X(F)$ . For the epsilon factor, it is more complicated.

Let  $\chi : CH_0(X) \rightarrow {}_1(X)^{\text{ab}}$  be the reciprocity map which is defined by sending the class  $[s]$  of a closed point  $s \in X$  to the geometric Frobenius  $\text{Frob}_s$ . Let  $G$  be a smooth sheaf on  $X$  and  $\det G : {}_1(X)^{\text{ab}} \rightarrow \times$  be the character associated to the determinant sheaf  $\det G$ . In joint work with N. Umezaki and Y. Zhao [NYZ17], we proved the following twist formula:

$$(X, F \otimes G) = \det G(-cc_X(F)) \cdot (X, F)^{\text{rank} G}, \quad (1.3.3)$$

which is conjectured by K. Kato and T. Saito in [KS08, Conjecture 4.3.11]. When  $F$  is the constant sheaf  $\mathbb{Z}$ , this is proved by S. Saito [SS84]. If  $F$  is a smooth sheaf on an open dense subscheme  $U$  of  $X$  such that the complement  $D = X \setminus U$  is a simple normal crossing divisor and the sheaf  $F$  is tamely ramified along  $D$ , then (1.3.3) is a consequence of [TS93, Theorem 1]. If  $\dim(X) = 1$ , the formula (1.3.3) follows from the product formula of Deligne and Laumon, c.f., [Del72, 7.11] and [Lau87, 3.2.1.1]. In [Vid09a, Vid09b], I. Vidal proved a similar result on a proper smooth surface over a finite field of characteristic  $p > 2$  under some technical assumptions.

As a corollary of (1.3.3), we proved the compatibility of characteristic class with proper push-forward by using the injectivity of the reciprocity map  $\chi$  [KS83, Theorem 1]. In general, T. Saito conjectures that the characteristic cycle (resp. characteristic class) should be compatible with proper push-forward, c.f., [TS16c, 7.2] and [TS16a, Conjecture 1].

**Problem 1.3.** Prove a similar formula of (1.3.3) if  $G$  is smooth only on an open dense subscheme  $U \subseteq X$  such that its wild ramification along  $X \setminus U$  is much smaller than that of  $F$ .

1.4. **Microlocal version of characteristic cycle and K-theory spectra.** In [Bei07], A. Beilinson developed the theory of topological epsilon factors using K-theory spectrum. Let  $R$  be a commutative ring. Let  $F$  be a perfect constructible complex of sheaves of  $R$ -modules on a compact real analytic manifold  $X$ . In [Bei07], he gave a Dubson-Kashiwara-style description of  $\det R(X, F)$ , and he asked that whether the construction in [Bei07] admits a motivic ( $\ell$ -adic or de Rham) counterpart. For de Rham cohomology, such a construction is given by his PhD student D. Patel in [Pat12, Pat17]. Based on [Pat17], T. Abe and D. Patel proved a similar twist formula in [AP17] for global de Rham epsilon factors in the classical setting of  $D_X$ -modules on smooth projective varieties over a field of characteristic zero. In principle, proving the formula at the level of K-theory spectra should also give formulas in higher K-theory. At the level of  $K_0$  (resp.  $K_1$ ), one gets formulas for the Euler characteristic (resp. determinants). It would be interesting to see the consequences at the level of  $K_2$  (or higher K-groups).

For  $\ell$ -adic cohomology, Beilinson's question is still open. For a constructible étale sheaf  $F$  on a smooth curve  $X$  over a finite field  $k$ , the precise statement for the  $\ell$ -factorization of  $\det(-\text{Frob}_k; R(X, F))$  was conjectured by P. Deligne [Del72] and proved by G. Laumon [Lau87] using local Fourier transform and  $\ell$ -adic version of principle of stationary phase.

**Problem 1.4.** For  $\ell$ -adic cohomology, can one give a  $\ell$ -factorization of  $\det R(X, F)$  for higher dimensional proper smooth varieties  $X$  over a finite field without using K-theory spectra [Bei07]? (Such a formula would imply (1.3.3) immediately. One can first consider this question for rank 1 clean sheaves (see also [AS06]).)

1.5. **Relation with other areas.** Geometric ramification theory can be applied to study many other problems. Here are three important applications among many others:

1.5.1. *Deligne's skeleton conjecture.* Using the cut-by-curve method and a little bit ramification theory, P. Deligne showed that there are finitely many irreducible smooth  $\mathbb{Q}$ -sheaves with bounded ramification, up to isomorphism and up to twist, on a smooth variety over a finite field. The details of his proof can be found in the article [EK12].

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Recently, T. Saito announced a proof for the geometric conductor formula assuming moreover that  $f$  is quasi-projective and is properly  $SS(F)$ -transversal over  $V \subseteq Y$ . Using this conductor formula, he proved the functoriality for proper push-forward of the characteristic cycles of constructible complexes for projective morphisms  $f: X \rightarrow Y$  of smooth schemes over a perfect field, under the assumption that  $\dim f_0 SS(F) \leq \dim Y$ .

1.5.3. *Swan class.* When they try to generalize the classical Grothendieck-Ogg-Shafarevich formula for curves to higher dimensional varieties, K. Kato and T. Saito [Sai01] define the Swan class  $Swan(f, \mathcal{F})$  as follows:  $Swan(f, \mathcal{F}) = \sum_{i \geq 0} (-1)^i \text{Tr}(f_0 R^i f_* \mathcal{F})$ .

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