A BLOW UP FORMULA FOR GYSIN PULL-BACK

NAOYA UMEZAKI, ENLIN YANG, AND YIGENG ZHAO

Abstract. In this note, we prove a blow up formula for Gysin pull-back of cycles by the zero section of a cotangent bundle (cf. Lemma 3.3). A special case of this formula is used in the proof of the twist formula for ε-factors [6].

1. Preliminaries on C-transversal condition

Definition 1.1. *Let X, Y and W be smooth schemes over a field k. We denote by $T^*_XX \subseteq T^*X$ the zero section of the cotangent bundle T^*X of X. Let C be a conical closed subset of T^*X, i.e., a closed subset which is stable under the action of the multiplicative group $\mathbb{G}_m$.*

(1) ([2, 1.2]) *Let $h\colon W \to X$ be a morphism over k. We say that h is C-transversal at $w \in W$ if the fiber $\big((C \times_X W) \cap dh^{-1}(T^*_WW)\big) \times_W w$ is contained in the zero-section $T^*_XX \times_X W \subseteq T^*X \times_X W$, where $dh\colon T^*X \times_X W \to T^*W$ is the canonical map. We say that h is C-transversal if h is C-transversal at any point of W.*

If h is C-transversal, we define $h^\circ C$ to be the image of $C \times_X W$ under the map $dh\colon T^*X \times_X W \to T^*W$. By [5, Lemma 3.1]*, $h^\circ C$ is a closed conical subset of T^*W.*

(2) ([5, Definition 7.1]) *Assume that X and C are purely of dimension d and that W is purely of dimension m. We say that a C-transversal map $h\colon W \to X$ is properly C-transversal if every irreducible component of $C \times_X W$ is of dimension m.*

(3) ([2, 1.2] *and* [5, Definition 5.3]) *We say that a morphism $f\colon X \to Y$ over k is C-transversal at $x \in X$ if the inverse image $df^{-1}(C) \times_X x$ is contained in the zero-section $T^*_YY \times_Y X \subseteq T^*Y \times_Y X$, where $df\colon T^*Y \times_Y X \to T^*X$ is the canonical map. We say that f is C-transversal if f is C-transversal at any point of X.*

1.2. Let X be a smooth scheme purely of dimension d over a field k. Let W be a smooth scheme purely of dimension m over k. Assume that $C \subseteq T^*X$ is a conical closed subset purely of dimension d. Let Z be a d-cycle supported on C and $h\colon W \to X$ a properly C-transversal morphism. Let $\mathrm{pr}_h\colon T^*X \times_X W \to T^*X$ be the first projection map. Since pr_h is a morphism between smooth schemes, the refined Gysin pull-back $\mathrm{pr}_h^! Z$ is well-defined in the sense of intersection theory [3, 6.6]. We define $h^*Z \in CH_m(h^\circ C)$ [5, Definition 7.1.2] to be

$$h^*Z := dh_*(\mathrm{pr}_h^! Z). \tag{1.2.1}$$

Notice that the push-forward is well-defined since $dh\colon T^*X \times_X W \to T^*W$ is finite on $C \times_X W$ by [2, Lemma 1.2 (ii)]. Since h is properly C-transversal, every irreducible component of $h^\circ C$ is of dimension m. Thus $CH_m(h^\circ C) = Z_m(h^\circ C)$. Hence we may regard h^*Z as a m-cycle on T^*W, which is supported on $h^\circ C$.

We prove the following commutative property for successively pull-backs.

Lemma 1.3. *Let X be a smooth scheme purely of dimension d over a field k. Consider the following commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{j} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{i} & X \end{array} \tag{1.3.1}$$

Date: August 6, 2018.
2010 *Mathematics Subject Classification.* Primary 14F20; Secondary 11G25, 11S40.

between equidimensional smooth schemes over k. Let $C \subseteq T^*X$ be a conical closed subset purely of dimension d. Assume that i and f are C-transversal, and g is $i^\circ C$-transversal. Let Z be a d-cycle supported on C. Then we have

(1) j is $f^\circ C$-transversal.

(2) $g^\circ i^\circ C = j^\circ f^\circ C \subseteq T^*U$ and an equality for cycle class $g^*i^*Z = j^*f^*Z$.

Proof. (1) This follows from [5, Lemma 3.4.3].

(2) We have a commutative diagram

(1.3.2)
$$\begin{array}{ccccc}
T^*X & \xleftarrow{\mathrm{pr}_f} & T^*X\times_X W & \xrightarrow{df} & T^*W \\
\uparrow{\scriptstyle \mathrm{pr}_i} & & \uparrow{\scriptstyle v} & \square & \uparrow{\scriptstyle \mathrm{pr}_j} \\
T^*X\times_X Y & \xleftarrow{u} & T^*X\times_X U & \xrightarrow{w} & T^*W\times_W U \\
\downarrow{\scriptstyle di} & \square & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle dj} \\
T^*Y & \xleftarrow[\mathrm{pr}_g]{} & T^*Y\times_Y U & \xrightarrow[dg]{} & T^*U
\end{array}$$

where the morphisms $\mathrm{pr}_f, \mathrm{pr}_g, \mathrm{pr}_i$ and pr_j are the first projections, df, dg, di, dj are morphisms induced from f, g, i, j respectively, and $v = \mathrm{id}\times j, u = \mathrm{id}\times g, r = di \times \mathrm{id}$. In the diagram (1.3.2), there are two Cartesian squares which are indicated by the symbols "$\square$". Then we have

(1.3.3)
$$\begin{aligned} g^\circ i^\circ C &= dg(\mathrm{pr}_g^{-1}(di(\mathrm{pr}_i^{-1}C))) = dg(r(u^{-1}(\mathrm{pr}_i^{-1}C))) \\ &= dg(r(v^{-1}(\mathrm{pr}_f^{-1}C))) = dj(\,!\,(v^{-1}(\mathrm{pr}_f^{-1}C))) \\ &= dj(\mathrm{pr}_j^{-1}(df(\mathrm{pr}_f^{-1}C))) = j^\circ f^\circ C. \end{aligned}$$

(1.3.4)
$$\begin{aligned} g^*i^*Z &= dg_*(\mathrm{pr}_g^!(di_*(\mathrm{pr}_i^!Z))) = dg_*(r_*(u^!(\mathrm{pr}_i^!Z))) \\ &= dg_*(r_*(v^!(\mathrm{pr}_f^!Z))) = dj_*(\,!_*(v^!(\mathrm{pr}_f^!Z))) \\ &= dj_*(\mathrm{pr}_j^!(df_*(\mathrm{pr}_f^!Z))) = j^*f^*Z \end{aligned}$$

where in (1.3.4) we used the push-forward formula [3, Theorem 6.2 (a)] and the fact that di (respectively df) is finite on $\mathrm{pr}_i^!Z$ (respectively $\mathrm{pr}_f^!Z$). This finishes the proof.

2. Localized Chern classes

2.1. Let X be a scheme of finite type over a field k, Z a closed subscheme of X and $U = X\setminus Z$. Let $\mathcal{K} = (\mathcal{K}_q, d_q)_q$ be a bounded complex of locally free $\mathcal{O}_X$ -modules of finite ranks such that $\mathcal{K}_q = 0$ for $q < 0$. Assume that the restriction $\mathcal{K}|_U$ is acyclic except at degree 0 and the cohomology sheaf $\mathcal{H}_0(\mathcal{K})|_U$ is locally free of rank $n-1$. Then for $i \geqslant n$, we have the so-called localized Chern class $c_i{}_Z^X(\mathcal{K}) \in CH^i(Z \to X)$ (cf. [1, Section 3], [3, Chapter 18] and [4, 2.3]). Consider the following ring (cf. [3, Chapter 17])

(2.1.1)
$$CH^*(Z\to X)^{(n)} = \prod_{i<n} CH^i(X\to X) \times \prod_{i\geqslant n} CH^i(Z \to X).$$

We regard the total localized Chern class $c_Z^X(\mathcal{K}) = ((c_i(\mathcal{K}))_{i<n}, (c_i{}_Z^X(\mathcal{K}))_{i\geqslant n})$ as an invertible element of $CH^*(Z\to X)^{(n)}$.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module such that the restriction $\mathcal{F}|_U$ is locally free of rank n. If $\mathcal{F}$ has a finite resolution $\mathcal{E}_\bullet \to \mathcal{F}$ by locally free $\mathcal{O}_X$-modules $\mathcal{E}_q$ of finite ranks, the localized Chern class $c_i{}_Z^X(\mathcal{F})$ for $i > n$ is defined as $c_i{}_Z^X(\mathcal{E}_\bullet)$. It is independent of the choice of a resolution.

2.2. The following Lemma 2.3 and Lemma 2.4 are slight generalizations of [4, Lemma 2.3.2] and [4, Lemma 2.3.4] respectively. We use the same arguments.

Lemma 2.3 ([4]). *Let X be a scheme of finite type over a field k. Let D be a Cartier divisor of X and $i\colon D \to X$ be the immersion. Let $\mathcal{E}$ be a locally free $\mathcal{O}_D$-module of rank n. Assume there exist a locally free $\mathcal{O}_X$-module $\widetilde{\mathcal{E}}$ of finite rank and a surjection $\widetilde{\mathcal{E}} \to i_*\mathcal{E}$ so that the localized Chern class $c_D^X(i_*\mathcal{E}(D)) \in CH^*(D\to X)^{(1)}$ is defined. We put $CH_*(X) = \oplus_i CH_i(X)$, $CH_*(D) = \oplus_i CH_i(D)$ and put $a_j(\mathcal{E}) = \sum_{k=j}^n \binom{k}{j} c_{n-k}(\mathcal{E}) \in CH^*(D\to D)$.*

(1) ([4, Lemma 2.3.2]) *For any invertible $\mathcal{O}_D$-module $\mathcal{L}$, we have*

$$\sum_{k=0}^{n} c_k(\mathcal{E}\otimes\mathcal{L}) = \sum_{j=0}^{n} a_j(\mathcal{E})c_1(\mathcal{L})^j: \tag{2.3.1}$$

(2) ([4, Lemma 2.3.2 and Corollary 2.3.2]) *For any $\ \in CH_*(X)$, we have equalities in $CH_*(D)$:*

$$(c_D^X(i_*\mathcal{E}(D)) - 1)\cap \ = c(\mathcal{E})^{-1}\sum_{j=1}^{n} a_j(\mathcal{E})D^{j-1}\cap i^! \ ; \tag{2.3.2}$$

$$(c_D^X(i_*\mathcal{O}_D)^{-1} - 1)\cap \ = -i^! \ : \tag{2.3.3}$$

where $i^!\colon CH_*(X)\to CH_*(D)$ denotes the Gysin map.

Proof. (1) See [4, Lemma 2.3.2].

(2) We use the same argument with [4, Lemma 2.3.2]. By deformation to the normal bundle, we may assume $X = \mathbb{P}^1_D$ is a $\mathbb{P}^1$-bundle over D and the immersion $i\colon D\to X$ is a section. Let $p\colon X\to D$ be the projection. Then $\mathcal{E} = i^*\mathcal{E}_X$ with $\mathcal{E}_X := p^*\mathcal{E}$. Since the map $i_* : CH_*(D)\to CH_*(X)$ is injective, it is reduced to the equalities for the usual Chern classes $c(i_*\mathcal{E}(D))$ and $c(i_*\mathcal{O}_D)$ by [4, Proposition 2.3.1.1]. By the exact sequence

$$0\to\mathcal{O}_X(-D)\to\mathcal{O}_X\to i_*\mathcal{O}_D\to 0; \tag{2.3.4}$$

we get $c(i_*\mathcal{O}_D)^{-1} = c(\mathcal{O}_X(-D))$. Thus $(c(i_*\mathcal{O}_D)^{-1}-1)\cap \ = -c_1(\mathcal{O}_X(D))\cap \ = -i^! \ $. This proves the equality (2.3.3). Now we prove (2.3.2). By the locally free resolution

$$0\to\mathcal{E}_X\to\mathcal{E}_X(D)\to i_*\mathcal{E}(D)\to 0; \tag{2.3.5}$$

we have

$$\begin{aligned} c(i_*\mathcal{E}(D)) - 1 &= c(\mathcal{E}_X)^{-1}(c(\mathcal{E}_X(D)) - c(\mathcal{E}_X)) \\ &\overset{(2.3.1)}{=} c(\mathcal{E})^{-1}\Big(\sum_{j=0}^{n} a_j(\mathcal{E})D^j - a_0(\mathcal{E})\Big) = c(\mathcal{E})^{-1}\sum_{j=1}^{n} a_j(\mathcal{E})D^j: \end{aligned} \tag{2.3.6}$$

Thus by the de nition of Gysin pull-back along a divisor [3, 2.6], we have

$$(c(i_*\mathcal{E}(D)) - 1)\cap \ = c(\mathcal{E})^{-1}\left(\sum_{j=1}^{n} a_j(\mathcal{E})D^j\right)\cap \ = c(\mathcal{E})^{-1}\left(\sum_{j=1}^{n} a_j(\mathcal{E})D^{j-1}\right)\cap i^! \tag{2.3.7}$$

Lemma 2.4 ([4, Lemma 2.3.4]). *Let X and C be regular schemes of nite type over a eld k. Let $i\colon C\to X$ be a closed immersion of codimension c with conormal sheaf $N_{C/X}$. Let $\ \colon X'\to X$ be the blow up of X along C, $\ _E\colon E = C\times_X X'\to C$ be the induced map and $i'\colon E\to X'$ be the closed immersion. We put*

$$(X;C) = \sum_{j=1}^{c} a_j(\ _E^* N_{C/X})E^{j-1} - \sum_{j=0}^{c} a_j(\ _E^* N_{C/X})E^j: \tag{2.4.1}$$

For any $\ \in CH_*(X')$, we have an equality in $CH_*(C)$:

$$\ _{E*}((c_E^{X'}(\ ^1_{X'/X}) - 1)\cap \ ) = c(N_{C/X})^{-1}\cap \ _{E*}(\ (X;C)\cap i'^! \ ); \tag{2.4.2}$$

If moreover $i'^! \ = \ _E^* \ $ for some $\ \in CH_*(C)$, then we have

$$\ _{E*}(\ (X;C)\cap i'^! \ ) = (-1)^c\cdot(c-1)\cdot \ ; \tag{2.4.3}$$

$$\ _{E*}((c_E^{X'}(\ ^1_{X'/X}) - 1)\cap \ ) = (-1)^c\cdot(c-1)\cdot c(N_{C/X})^{-1}\cap \ : \tag{2.4.4}$$

Proof. Note that the canonical map $\ ^1_{X'/X}\to i'_* \ ^1_{E/C}$ is an isomorphism. Since $E = \mathbb{P}((N_{C/X})^\vee)$ is a $\mathbb{P}^{c-1}$-bundle over C, we have an exact sequence $0\to \ ^1_{E/C}\to \ _E^* N_{C/X}(-1)\to\mathcal{O}_E\to 0$. Hence, we have $c_E^{X'}(\ ^1_{X'/X}) = c_E^{X'}(i'_* \ _E^* N_{C/X}(-1))c_E^{X'}(i'_*\mathcal{O}_E)^{-1}$. By the exact sequence $0\to\mathcal{O}_{X'}(-E)\to\mathcal{O}_{X'}\to i'_*\mathcal{O}_E\to 0$, we get

$$0\to \ _E^* N_{C/X}\to \ _E^* N_{C/X}(E)\to \ _E^* N_{C/X}(E)\to 0: \tag{2.4.5}$$

By Lemma 2.3, we have

$$(2.4.6)\qquad \begin{aligned}&(c_E^{X'}(i'_* \ {}^*_E N_{C/X}(-1))c_E^{X'}(i'_*\mathcal{O}_E)^{-1} - 1)\cap \\ &= (c_E^{X'}(i'_* \ {}^*_E N_{C/X}(-1)) - 1)\cap \quad + c_E^{X'}(i'_* \ {}^*_E N_{C/X}(-1))(c_E^{X'}(i'_*\mathcal{O}_E)^{-1} - 1)\cap \\ &\overset{(2.3.3)}{=} (c_E^{X'}(i'_* \ {}^*_E N_{C/X}(-1)) - 1)\cap \quad - c(\ {}^*_E N_{C/X}(-1))\cap i'^! \\ &\overset{(2.4.5)}{=} (c_E^{X'}(i'_* \ {}^*_E N_{C/X}(E)) - 1)\cap \quad - c_E(\ {}^*_E N_{C/X})^{-1} c_E(\ {}^*_E N_{C/X}(E))\cap i'^! \\ &\overset{(2.3.2)}{=} c_E(\ {}^*_E N_{C/X})^{-1}\left(\sum_{j=1}^{c} a_j(\ {}^*_E N_{C/X})E^{j-1}\cap i'^! \quad - \sum_{j=0}^{c} a_j(\ {}^*_E N_{C/X})E^{j}\cap i'^! \quad \right) \\ &= c(N_{C/X})^{-1}\cap \quad {}_{E*}(\quad (X;C)\cap i'^! \quad):\end{aligned}$$

By [3, Remark 3.2.4, p.55], we have $E^c = -\sum_{j=1}^{c} c_j(\ {}^*_E N_{C/X})E^{c-j}$. Assume $i'^! \quad = \ {}^*_E$ for some $\in CH_*(C)$. By [3, Proposition 3.1 (a)], we have ${}_{E*}(E^j\cap \ {}^*_E \ ) = 0$ for $j < c-1$ and ${}_{E*}(E^{c-1}\cap \ {}^*_E \ ) = (-1)^{c-1}$. Substituting these identities, we have

$$(2.4.7)\qquad \begin{aligned}&{}_{E*}(\quad (X;C)\cap i'^! \quad) \\ &= (-1)^{c-1}\cdot(a_c(N_{C/X}) - a_{c-1}(N_{C/X}) + a_c(N_{C/X})c_1(N_{C/X}))\cap\end{aligned}$$

Since $a_c(N_{C/X}) = 1$, $a_{c-1}(N_{C/X}) = c + c_1(N_{C/X})$ then ${}_{E*}(\quad (X;C)\cap i'^! \quad) = (-1)^c\cdot(c-1)\cdot$ and ${}_{E*}((c_E^{X'}(\ {}^1_{X'/X}) - 1)\cap \quad) = (-1)^c\cdot(c-1)\cdot c(N_{C/X})^{-1}\cap$.

3. Blow up formula for Gysin pull-back

3.1. Let X be a smooth scheme purely of dimension d over a eld k. We denote by $0_X: X \to T^*X$ the zero section of the cotangent bundle T^*X. We denote by $0_X^! \in CH^d(X \to T^*X)$ the (re ned) Gysin map [3, 6.2], where $CH^d(X \to T^*X)$ is the bivariant Chow group [3, De nition 17.1]).

3.2. We recall a method for calculating the Gysin map $0_X^!$ by using Chern classes. Let X be a regular scheme separated of nite type over a eld. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-modules of rank d on X. Let $E = \mathrm{Spec}(\mathrm{Sym}^\bullet_{\mathcal{O}_X}\mathcal{E}^\vee)$ be the associated vector bundle of rank d on X with structure morphism $: E \to X$. The projective bundle of E is $\mathbb{P}(E) = \mathrm{Proj}(\mathrm{Sym}^\bullet_{\mathcal{O}_X}\mathcal{E}^\vee)$. We have a closed immersion $\mathbb{P}(E) \hookrightarrow P(E\oplus 1) := P(E\oplus\mathbb{A}^1_X)$ with open complementary $E \hookrightarrow \mathbb{P}(E\oplus 1)$. Let $s: X \to E$ be the zero section. Let $k \geqslant 0$ be an integer and $\in CH_k(E)$. For any element $\in CH_k(\mathbb{P}(E\oplus 1))$, if the restriction of to $CH_k(E)$ equals to , then we have [3, Proposition 3.3]

$$(3.2.1)\qquad s^!(\ ) = q_*(c_d(\ )\cap \ );$$

where $= \frac{q^*(\mathcal{E}\oplus 1)}{\mathcal{O}_{\mathbb{P}(E\oplus 1)}(-1)}$ is the universal rank d quotient bundle of $q^*(\mathcal{E}\oplus 1)$. For any element $\in CH_*(X) = \oplus_i CH_i(X)$, we denote by $\{ \ \}_j$ the dimension j part of , i.e., the image of by the projection $CH_*(X) \to CH_j(X)$. Let $c(\ )$ be the total Chern class of , then we can write (3.2.1) as follows

$$(3.2.2)\qquad s^!(\ ) = \{q_*(c(\ )\cap \ )\}_{k-d}:$$

By the Whitney sum formula for Chern classes [3, Theorem 3.2], we have

$$(3.2.3)\qquad c(\ ) = c(q^*\mathcal{E})\cdot c(\mathcal{O}_{\mathbb{P}(E\oplus 1)}(-1))^{-1}:$$

Thus the formula (3.2.2) can be written in the following way

$$(3.2.4)\qquad \begin{aligned}s^!(\ ) &= \{q_*(c(q^*\mathcal{E})\cap c(\mathcal{O}_{\mathbb{P}(E\oplus 1)}(-1))^{-1}\cap \ )\}_{k-d} \\ &= \{c(\mathcal{E})\cap q_*(c(\mathcal{O}_{\mathbb{P}(E\oplus 1)}(-1))^{-1}\cap \ )\}_{k-d}\end{aligned}$$

where the last equality follows from the projection formula [3, Theorem 3.2].

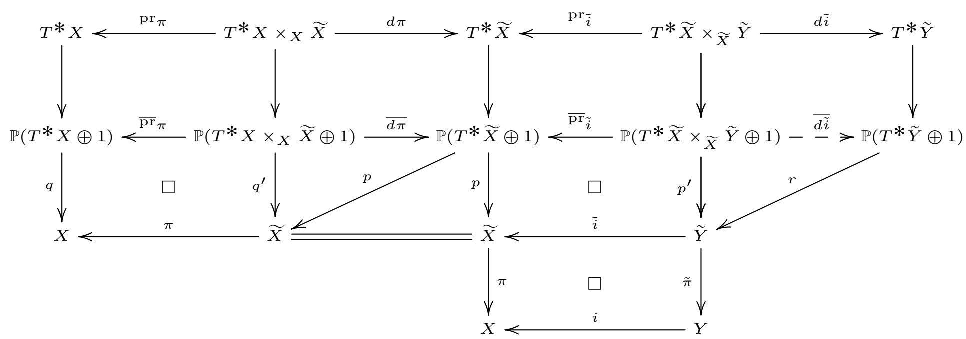
Lemma 3.3. *Let X and Y be smooth and connected schemes over a eld k and let $i: Y \hookrightarrow X$ be a closed immersion of codimension c. Let $: \tilde{X} \to X$ be the blow up of X along Y. Let $C \subseteq T^*X$ be a conical* *closed subset purely of dimension $d = \dim X$ and let Z be a d-cycle supported on C. Suppose π and i are properly C-transversal. Then we have an equality in $CH_0(X)$:*

$$\pi_*(0^!_{\widetilde{X}}(\pi^*Z)) = 0^!_X(Z) + (-1)^c \cdot (c-1) \cdot i_*(0^!_Y(i^*Z)); \tag{3.3.1}$$

For the definition of the Gysin map $0^!_\bullet$, see Subsection 3.1.

Proof. Let $\widetilde{Y}$ be the exceptional divisor of $\pi\colon \widetilde{X} \to X$ with projection map $\tilde{\pi}\colon \widetilde{Y} \to Y$. Let $\tilde{i}\colon \widetilde{Y} \hookrightarrow \widetilde{X}$ be the closed immersion. We have a commutative diagram

(3.3.2)



where pr_π and $\mathrm{pr}_{\tilde{i}}$ are the first projections, $d\pi\colon T^*X \times_X \widetilde{X} \to T^*\widetilde{X}$ (respectively $d\tilde{i}\colon T^*\widetilde{X} \times_{\widetilde{X}} \widetilde{Y} \to T^*\widetilde{Y}$) is the map induced by $\pi\colon \widetilde{X} \to X$ (respectively $\tilde{i}\colon \widetilde{Y} \to Y$), the maps $\overline{\mathrm{pr}}_\pi$, $\overline{d\pi}$, $\overline{\mathrm{pr}}_{\tilde{i}}$ and $\overline{d\tilde{i}}$ are the maps induced by pr_π, $d\pi$, $\mathrm{pr}_{\tilde{i}}$ and $d\tilde{i}$ respectively, all other maps are either the canonical projection morphisms or open immersions. In (3.3.2), we use the symbol "$\square$" to mean the square is a Cartesian diagram. For example, the most left-bottom square in (3.3.2) is Cartesian since q' is proper and $\mathbb{P}(T^*X \times_X \widetilde{X} \oplus 1)$ has dense image in $\mathbb{P}(T^*X \oplus 1) \times_X \widetilde{X}$. Note also that the map $\overline{d\tilde{i}}$ is only well-defined on the open subscheme $T^*\widetilde{X} \times_{\widetilde{X}} \widetilde{Y}$, but this is enough for our purpose (cf. [5, Lemma 6.4]).

For any $\alpha \in CH_d(T^*X)$, we denote by $\overline{\alpha} \in CH_d(\mathbb{P}(T^*X \oplus 1))$ an extension of α (cf. 3.2). We choose an extension $\overline{Z} \in CH_d(\mathbb{P}(T^*X \oplus 1))$ of Z. Then $\overline{\mathrm{pr}}^!_\pi(\overline{Z})$ is an extension of $\mathrm{pr}^!_\pi Z$. Since π is C-transversal, the push-forwards $d\pi_*(\mathrm{pr}^!_\pi Z)$ and $\overline{d\pi}_*(\overline{\mathrm{pr}}^!_\pi(\overline{Z}))$ are well-defined, and $\overline{d\pi}_*(\overline{\mathrm{pr}}^!_\pi(\overline{Z}))$ is an extension of $d\pi_*(\mathrm{pr}^!_\pi Z)$.

Since $\tilde{\pi}$ is smooth, thus $\tilde{\pi}$ is $i^\circ C$-transversal by [5, Lemma 3.4.1]. By Lemma 1.3, $\tilde{i}$ is $\pi^\circ C$-transversal and we have

$$\tilde{\pi}^* i^* Z = \tilde{i}^* \pi^* Z. \tag{3.3.3}$$

The following exact sequence ($\tilde{i}_* \Omega^1_{\widetilde{Y}/Y} \simeq \Omega^1_{\widetilde{X}/X}$)

$$0 \to \pi^*\Omega^1_X \to \Omega^1_{\widetilde{X}} \to \tilde{i}_*\Omega^1_{\widetilde{Y}/Y} \to 0 \tag{3.3.4}$$

gives a resolution of $\tilde{i}_*\Omega^1_{\widetilde{Y}/Y}$ by locally free sheaves of finite rank. Thus the localized Chern class $c_k{}^X_Y(\tilde{i}_*\Omega^1_{\widetilde{Y}/Y}) \in CH^*(Y \to X)$ is well-defined for $k \geqslant 1$ (cf. Subsection 2.1). In order to simplify the notation, we put $c^{\mathrm{loc}}_k(\tilde{i}_*\Omega^1_{\widetilde{Y}/Y}) := c_k{}^X_Y(\tilde{i}_*\Omega^1_{\widetilde{Y}/Y})$ and $c^{\mathrm{loc}}_k(p^*\tilde{i}_*\Omega^1_{\widetilde{Y}/Y}) := c_k{}^{\mathbb{P}(T^*\widetilde{X}\oplus 1)}_{\mathbb{P}(T^*\widetilde{X}\times_{\widetilde{X}}\widetilde{Y}\oplus 1)}(p^*\tilde{i}_*\Omega^1_{\widetilde{Y}/Y})$. Similarly, we denote by c^{loc} the total localized Chern class. Applying the Whitney sum formula for (localized) Chern classes (cf. [1, Proposition 3.1]) to the exact sequence (3.3.4), we get

$$c(\Omega^1_{\widetilde{X}}) = c(\pi^*\Omega^1_X) \cdot c^{\mathrm{loc}}(\tilde{i}_*\Omega^1_{\widetilde{Y}/Y}) = c(\pi^*\Omega^1_X) + c(\pi^*\Omega^1_X) \cdot (c^{\mathrm{loc}}(\tilde{i}_*\Omega^1_{\widetilde{Y}/Y}) - 1): \tag{3.3.5}$$

We will simply denote by $\mathcal{O}(1)$ for $\mathcal{O}_{\mathbb{P}(T^*\widetilde{X}\oplus 1)}(1)$ (and also for $\mathcal{O}_{\mathbb{P}(T^*X\oplus 1)}(1)$ and so on) in the following calculations. We have

$$(3.3.6) \qquad {}_*(0^!_{\widetilde{X}}(\ {}^*Z)) \overset{(3.2.4)}{=} \Big\{\ {}_*\Big(c(\ {}^1_{\widetilde{X}}) \cap p_*\big(c(\mathcal{O}(-1))^{-1} \cap \overline{d}\ {}_*\overline{\mathrm{pr}}^!\,\overline{Z}\big)\Big)\Big\}_0$$

$$\overset{(3.3.5)}{=} \Big\{\ {}_*\Big(c(\ {}^*\ {}^1_X)\cdot c^{\mathrm{loc}}(\check{T}_*\ {}^1_{\tilde{Y}/Y}) \cap p_*\big(c(\mathcal{O}(-1))^{-1} \cap \overline{d}\ {}_*\overline{\mathrm{pr}}^!\,\overline{Z}\big)\Big)\Big\}_0$$

$$\overset{(a)}{=} \Big\{c(\ {}^1_X) \cap\ {}_*\Big(c^{\mathrm{loc}}(\check{T}_*\ {}^1_{\tilde{Y}/Y}) \cap p_*\big(c(\mathcal{O}(-1))^{-1} \cap \overline{d}\ {}_*\overline{\mathrm{pr}}^!$$

Consider the following commutative diagram induced from the morphism $\sim : \widetilde{Y} \to Y$.

$$(3.3.10) \qquad \begin{array}{ccccc}
T^*Y & \xleftarrow{\mathrm{pr}_{\tilde{\pi}}} & T^*Y \times_Y \widetilde{Y} & \xrightarrow{d\sim} & T^*\widetilde{Y} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{P}(T^*Y \oplus 1) & \xleftarrow{\overline{\mathrm{pr}}_{\tilde{\pi}}} & \mathbb{P}(T^*Y \times_Y \widetilde{Y} \oplus 1) & \xrightarrow{\overline{d\sim}} & \mathbb{P}(T^*\widetilde{Y} \oplus 1) \\
\downarrow s & \Box & \downarrow s' & \swarrow r & \\
Y & \xleftarrow{\sim} & \widetilde{Y} & & \\
\downarrow i & \Box & \downarrow \tilde{\imath} & & \\
X & \longleftarrow & \widetilde{X} & &
\end{array}$$

Since $\overline{d\sim}_*\overline{\mathrm{pr}}^! \ \overline{i^*Z}$ is an extension of $\sim^* i^*Z$ to $\mathbb{P}(T^*\widetilde{Y} \oplus 1)$, thus (3.3.9) equals to

$$(3.3.11) \qquad \tilde{\imath}^! \left(p_* \left(c(\mathcal{O}(-1))^{-1} \cap \overline{d\ }_* \overline{\mathrm{pr}}^! \ \overline{Z}\right)\right) = r_* \left(c(\mathcal{O}(-1))^{-1} \cap \overline{d\sim}_* \overline{\mathrm{pr}}^!_{\sim} \overline{i^*Z}\right)$$
$$\overset{(c)}{=} r_* \overline{d\sim}_* \left(c(\mathcal{O}(-1))^{-1} \cap \overline{\mathrm{pr}}^!_{\sim} \overline{i^*Z}\right) = r_* \overline{d\sim}_* \left(\overline{\mathrm{pr}}^* c(\mathcal{O}(-1))^{-1} \cap \overline{\mathrm{pr}}^!_{\sim} \overline{i^*Z}\right)$$
$$\overset{(d)}{=} r_* \overline{d\sim}_* \overline{\mathrm{pr}}^!_{\sim} \left(c(\mathcal{O}(-1))^{-1} \cap \overline{i^*Z}\right)$$

where we used the projection formula [3, Theorem 3.2] in step (c), and (d) follows from [3, Proposition 17.3.2]. By the commutative diagram (3.3.10) and the push-forward formula [3, Theorem 6.2], we have

$$(3.3.12) \qquad r_* \overline{d\sim}_* \overline{\mathrm{pr}}^!_{\sim} = s'_* \overline{\mathrm{pr}}^!_{\sim} = \sim^! s_* = \sim^* s_*.$$

By (3.3.11) and (3.3.12), the second term of (3.3.7) equals to

$$(3.3.13) \qquad \left\{ i_* \left(c(i^* \ {}^1_X) \cap c(N_{Y/X})^{-1} \cap \sim_* \left(\ (X;Y) \cap \tilde{\imath}^! \left(p_* \left(c(\mathcal{O}(-1))^{-1} \cap \overline{d\ }_* \overline{\mathrm{pr}}^! \ \overline{Z}\right)\right)\right)\right)\right\}_0$$
$$= \left\{ i_* \left(c(i^* \ {}^1_X) \cap c(N_{Y/X})^{-1} \cap \sim_* \left(\ (X;Y) \cap \sim^* s_* \left(c(\mathcal{O}(-1))^{-1} \cap \overline{i^*Z}\right)\right)\right)\right\}_0$$
$$\overset{(2.4.3)}{=} (-1)^c \cdot (c-1) \left\{ i_* \left(c(i^* \ {}^1_X) \cap c(N_{Y/X})^{-1} \cap s_* \left(c(\mathcal{O}(-1))^{-1} \cap \overline{i^*Z}\right)\right)\right\}_0$$
$$\overset{(1)}{=} (-1)^c \cdot (c-1) \left\{ i_* \left(c(\ {}^1_Y) \cap s_* \left(c(\mathcal{O}(-1))^{-1} \cap \overline{i^*Z}\right)\right)\right\}_0$$
$$\overset{(3.2.4)}{=} (-1)^c \cdot (c-1) \cdot i_* 0^!_Y (i^*Z).$$

where the step (1) follows from $c(i^* \ {}^1_X) \cdot c(N_{Y/X})^{-1} = c(\ {}^1_Y)$ since we have an exact sequence

$$(3.3.14) \qquad 0 \to N_{Y/X} \to i^* \ {}^1_X \to \ {}^1_Y \to 0;$$

where $N_{Y/X}$ is the conormal sheaf associated to the the regular immersion $i : Y \to X$.

Finally, by (3.3.6), (3.3.8) and (3.3.12), we get (3.3.1). This nishes the proof.

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Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan

E-mail address: umezaki@ms.u-tokyo.ac.jp, umezakinaoya@gmail.com

Fakultat fur Mathematik, Universitat Regensburg, 93040 Regensburg, Germany

E-mail address: enlin.yang@mathematik.uni-regensburg.de, yangenlin0727@126.com

Fakultat fur Mathematik, Universitat Regensburg, 93040 Regensburg, Germany

E-mail address: yigeng.zhao@mathematik.uni-regensburg.de