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motive 2019-2023 [UYZ20, YZ21, JY21] [YZ22, JY22, JSY22, XY23]

1. [UYZ20] ε Kato Saito [KS08, Conjecture 4.3.11] [1]
2. [YZ21] Kato-Saito Abbes Saito transversal [LZ22] ULA universally locally acyclic sheaves
3. [YZ22] non-acyclicity NA Saito Saito Abbes-Saito NA Milnor
4. [XY23] NA Milnor NA
5. [JY21, JY22, JSY22] motive NA Artin Grothendieck-Ogg-Shafarevich

- 2022 2026 L 300
- 2023 2026 - 45
- 2020 2022 - ℓ- ε 27

1 2 Kato-Saito ε 3 NA Saito 4 NA 5 motive

1

characteristic class characteristic cycle

1.1 S Sch_S S Λ S $X \in \mathrm{Sch}_S$ $D_{\mathrm{ctf}}(X, \Lambda)$ X tor

[1] Swan

Λ-

1.2 S

(G) $S = \mathrm{Spec} k$ k $p > 0$ perfect field $\mathcal{K} \in D_{\mathrm{ctf}}(S, \Lambda)$ $\dim\mathcal{K} = \mathrm{rank}\mathcal{K}$

(A) S η S generic point s $k(s)$ $p > 0$ $\mathcal{K} \in D_{\mathrm{ctf}}(S, \Lambda)$ Swan $\mathrm{Sw}\mathcal{K}$ $\mathrm{dimtot}\mathcal{K} = \dim\mathcal{K}_{\bar{\eta}} + \mathrm{Sw}\mathcal{K}$ Artin $a_S(\mathcal{K}) = \dim\mathcal{K}_{\bar{\eta}} - \dim\mathcal{K}_{\bar{s}} + \mathrm{Sw}\mathcal{K} = \mathrm{dimtot}R\Phi_{\mathrm{id}}(\mathcal{K})$ $R\Phi$

Artin Swan Abbes Saito [AS07] Kato Saito [KS08, KS12] Swan Saito [Sai17] Beilinson Riemann-Roch

1.3 $f : X \to S$ $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$

- (G) $\chi_c(X_{\bar{k}}, \mathcal{F}) = \dim Rf_!\mathcal{F}$
- (A) $\mathrm{Sw}Rf_!\mathcal{F}$ $\mathrm{dimtot}Rf_!\mathcal{F}$ $a_S(Rf_!\mathcal{F})$

Abbes [Abb00] Abbes-Saito [AS07] Bloch [Blo87] Deligne [Del72, Del11] Hu [Hu15] Kato-Saito [KS04, KS08, KS12] Laumon [Lau83] Saito [Sai17, Sai18, Sai21] Tsushima [Tsu11]

Swan Kato Saito [KS04, KS08, KS12] K- Swan Swan Abe [Abe21] /

[YZ22] NA non-acyclicity locus Z NA Z X $X \setminus Z \to S$ $\mathcal{F}|_{X\setminus Z}$ ULA NA Artin /motivic (3.9.1) motive Riemann-Roch [JY22] .

1.4 Grothendieck-Ogg-Shafarevich (G) X $S = \mathrm{Spec}k$ Euler-Poincaré $\chi(X_{\bar{k}}, \mathcal{F})$ Grothendieck-Ogg-Shafarevich (GOS)

(1.4.1) $$\chi(X_{\bar{k}}, \mathcal{F}) = \dim\mathcal{F}_{\bar{\eta}} \cdot \chi(X_{\bar{k}}, \Lambda) - \sum_{x\in Z} a_x(\mathcal{F}) \cdot \deg(x),$$

η X Z $\mathcal{F}|_{X\setminus Z}$ $a_x(\mathcal{F}) = a_{X_{(x)}}(\mathcal{F})$ $\mathcal{F}$ x Artin Gauss-Bonnet-Chern $\chi(X_{\bar{k}}, \Lambda) = \deg(c_1(\Omega^{1,\vee}_{X/k}) \cap [X])$ GOS (1.4.1)

(1.4.2) $$\chi(X_{\bar{k}}, \mathcal{F}) = \deg(cc_{X/k}(\mathcal{F})),$$

$cc_{X/k}(\mathcal{F})$ X 0

(1.4.3) $$cc_{X/k}(\mathcal{F}) = \dim \mathcal{F}_{\bar{\eta}} \cdot c_1(\Omega^{1,\vee}_{X/k}) \cap [X] - \sum_{x \in Z} a_x(\mathcal{F}) \cdot [x] \quad \text{in} \quad \mathrm{CH}_0(X).$$

1.5 [2] GOS [KS90]
Kashiwara Schapira microlocal
$\mathcal{D}$ [Bei07] Beilinson Kashiwara-Schapira
ℓ de Rham Deligne
irregular singularity
[Del11] Deligne Beilinson
singular support [Bei16] Deligne Saito [Sai17, Theorem 4.9 and Theorem 6.13]

X k n $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$
$SS(\mathcal{F})$ T^*X X $f \colon X \to \mathbb{A}^1_k$ df
$SS(\mathcal{F})$ f $\mathcal{F}$ ULA Beilinson [Bei16] $SS(\mathcal{F})$
n Saito [Sai17] $\mathbb{Z}$ $SS(\mathcal{F})$ n $CC(\mathcal{F})$
$CC(\mathcal{F})$

(a) ( ) X k [3]

(1.5.1) $$\chi(X_{\bar{k}}, \mathcal{F}) = (CC(\mathcal{F}), T^*_X X)_{T^*X},$$

$T^*_X X$ T^*X

(b) (Milnor) $f : X \to C$ X C x_0 f
$SS(\mathcal{F})$ isolated characteristic point ([Sai17, Definition 3.7])

(1.5.2) $$-\mathrm{dimtot}\, R\Phi_f(\mathcal{F})_{\bar{x}_0} = (CC(\mathcal{F}), df)_{T^*X, x_0}.$$

(c) ( ) X Y η Y $\mathcal{F} \in D^b_c(X, \Lambda)$ $f : X \to Y$ f $\mathcal{F}$ f $V \subseteq Y$
properly $SS(\mathcal{F})$ $y \in Y$ ( )

(1.5.3) $$-a_y(Rf_*\mathcal{F}) = (CC(\mathcal{F}), df)_{T^*X, y},$$

$a_y(Rf_*\mathcal{F}) = \chi(X_{\bar{\eta}}, \mathcal{F}) - \chi(X_{\bar{y}}, \mathcal{F}) + \mathrm{Sw}_y R\Gamma(X_{\bar{\eta}}, \mathcal{F})$ Artin

$\mathcal{F}$ Λ $CC(\Lambda) = (-1)^n \cdot [T^*_X X]$ (1.5.1) Gauss-Bonnet-Chern $\chi(X_{\bar{k}}, \Lambda) = \deg(c_n(\Omega^{1,\vee}_{X/k}) \cap [X])$ (1.5.2) Milnor ([Del73, Conjecture 1.9, P200])

(1.5.4) $$-\mathrm{dimtot}\, R\Phi_f(\Lambda)_{\bar{x}_0} = (-1)^n \cdot \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{E}xt^1(\Omega^1_{X/C}, \mathcal{O}_X))_{x_0},$$

(1.5.3) Bloch ([Blo87])
$\mathcal{F}$ tamely (1.5.2) Milnor ([Y14, 1.12])
X (1.5.1) Grothendieck-Ogg-Shafarevich (1.4.2)
X Deligne Laumon "non-fierce"

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[2] Swan Kato Saito [KS08, KS12] GOS 2.4

[3] Abe

K4 >/ D X @

7 D N H X7 H K L 7 D N@f ¼ 6 D L p M,`K4,º 6 ` ‰ ¸¹ D#¾ gK4,º vG " †,º 0 L O Q R U 2± Á ¢ E ? © >

p ¼ D5#,º ï õ ¸ G " K › A Ð • Æ

Ð • 6 D L ' H I L Q L W L R Q 0 X : X !@ T X pM, `M˜ > Ð • F ,º (‹) /p]2± p A F&M,5*2± Æ

cc X / k (F) = 0 ¦ X (CC(F)) L Q &(X):

!™ O û ˜ 69<‡1Í e û ë ´ $µ> G " }Aô S e$- >(¯ ·,º > G " 'I ? ' H O L J Q H 0 L O Q R U ¢ E,º)R) Â % O R,º F 2 K ¢ E)R) ¸F Z ` ¢ E ^ 68b û ë A 'I"× ?>á@ é > ^ >2 U J 7 K p R U q P H ¸ 2 U@ R JÄ÷]D R ¼ % O R,º F 2 K ¢ E;ª a ¼ 0 L O Q R U ¢ E b ^ >. 6 & R U R O O D U \ @ . D W Ô R 6 D L A÷ D R ¼ 2 † ¢ E e 4ÜL#4Ú5* , › } 8&ï,º p#¾@ }Aô,º ÞAð >(¯ a f ¸ ‰-. /5* ¦1• Ä L , ¸ 0 L O Q R U ¢ E)R) e!™.¤,º >

6 ` ¾8 'I,ºL$NÎ e ¸ e \ Ž ^1Í e(~ b,º = = ¢ E > ^ >$ E E @ ¸ $ E E A÷ D ¼ vG @ (4õ,º1Í e * U R W K H Q G L H F N 2 ¢ E >6 K D W D R 6 D H Y W > K 6 @ Í f ¼ 6 ` Q 5*(~ b > ^ >6 D L @ ¸ 6 D L D W R ¼!™ O ¸ ˜,º) : • = Q >FPEý µ+^ @ BB9 ` ` — , ¸ F K › ¼!™ O ¸ ˜ @ % ºFV © ‹ x,º } 8 e Ç ´ |"× ?A÷ D ¬ Ž ^] µ> G " U » ¸$- >(¯ · û ˜ A,º 0 L O Q R U ¢ E D % O R F 2 K ¢ E % +^) : • = QH æ>žF& 0 › > / Ä1Í e (M˜ ¸ 2 R H f ¼G <5 Ò >2 R H@^ ` ›.J0¬A× H , c ø?÷1b " "8¸ ¸FPEý ºFV 1 $2±,º1Í e(~ b ¸ G " < _+^ @ BB9 ï# ›.J0¬!šL$NÎ >

/p]2± D " †

" † L 3 ¦ Â " † e M ¦ ¯4èN¼ Â s5•+ >ž/pAð c,º Z `H ?· /B— > <!•*<Að DF Z `H •L* ?H ?·,º ©31 >

Aôk e(¯ · p p ,º ?L† ¸ f : X ! 6 S (H) F e-. /5* ¦ p n ,º •% : § 7 : >Aô e(¯ · p ` 6 p ,º ?L† > / X @ 6 ` % ºFV © ‹ WF ¸Aô D(F) e F ,º / ¬ RHom(F ; K X) ¸ ¬ c K X = Rf ! e / ¬ C ˜ > L 3 ¦ L(X; F ; t) % Cé A 3 ¦ ï0A

L(X; F ; t) = "(X; F) t (X k ;F) L(X; D (F); t ¹);

¬ c

"(X; F) = G H W U R,E (X k ; F)) ¹

e " † 3 ¦ ï0A ,º n ¦N¯ ¸6B (X k ; F) e F ,º (X O H U 3 R L Q F D U p) ¦ ï0A

c ¸ † (X k ; F) Â "(X; F) G3 D <!•*<Að ? © > ÔL{ @ ¸ (X k ; F) = G H t c X (F) ø?÷ D >" † D <!•*<Að,º ©31 O * Ö C x >

£5 ¢ E Aô X : CH 0 (X) ! 1 (X) D E e È Á V : ¸ ¹ < 6 `L#&ï s 2 X v Ð •,º2± [s] V : f) U R´E ‹) U R E H µ Q A ô X G e X @,º •% x ¸0 p ,º>, M E x G H 6 W 0 ¼>ž/p G H W 1 (X) D E ! > ^ D 8 P H] D N L « Ð,º > ' c >8 < = @G "A÷ D ¼ " † £5 ¢

E Æ

" (X; F G) = G G(Wcc$_X$ (F)) " (X; F) U DQ N

B ¢ E e . D W R 6 D L W R 6 & R Q M H F W X U H) R) ¢ E @% F e n r x , F

^ 6 D L @ Í f ¼A÷ D > ¸ Ò F e X ,º 6 ` 60V ü † ¸ ˜ U @,º •% x ¸ J>›Lü D = X nU

e 6 `1¶ ‹!™?ú Ú ÿLš † 6 1 & '¸J x F "õEï+, D eP¥ <!•,º FÙ ~ £5 ¢ E e 6 D L

7 K H R U H P ` Þ@ð > ¸ Ò G L (X) = 1 ¸ ¢ E % ' H O L J Q H X P R Q % ¢ E Í

› ø?÷ H O H D X > /@Ä(¯ · C p ,º ?L† @,º3] > •% (M˜ ¸ ^ Ñ

¶ e] }Aô — , A ¸ 9 L G D 9 L G 9 L G @A÷ D ¼2± r,º5 Ò >

' p £5 ¢ E ,º ÞAð ¸_+^ È Á V : X e ‹ : 6 7 K H R U H P A÷ D ¼/p

]2± D3] > Þ 0 e-. ï,º > 68b6B@6 ¸ ^ ` ¾ g,º }Aô — , A ¸ 6 D L W R ¼/p]K4 D3] > Þ

0 e-. ï,º ø?÷ 6 D L 6 D L & R Q M H F W X U H 7 K H R U H P > ^Að ½@ < = @ ¸

G " 0 ¼ £5 ¢ E,º-. /(~ b >

L$NÎ ¸ Ò G û ^ X ,º `0V ü 6 † ¸ ˜ U X @ e •% ,º ¸ | ¬"õ-vEï+, X nU ,ºH <

!•F E Ä F ,ºH <!• ¸ e \ Ž ^ 6 `2± r,º £5 ¢ E Ë

6 Z D Q p ¼ < (4õ @,º * U R W K H Q G L H F N 2 ¢ E Þ6µK Q 5 D L H @ L F K . D W R

6 D L W R 6 @ KF ¼ 6 Z D Q > 6 D L W R R#• ¸B 6 Z D Q Ê ‰ % +^/p]K45 0 › ø?÷

6 D L & R Q M H F W X U H B* ¸A@ X e(¯ · p p > 0 ,º Â% k @,º •% ¸ ˜ ¸ X e X

,º •% 3] L > / Ä X @,º •% % ºFV © ‹ x F ¸ 6 D L W R •F ,º 6 Z D Q - ? ª31 ¦ ¸ ¬ J1• Ä

CC(j$_!$F) U D Q N C(j$_!$) FPEýM, `M˜,º ÿ > / Ä ?L† @,º •% (M˜ ¸ G " ^ 8 < = @

A÷ D ¼ 6 D L W R),º g(~ b > ¸ Ò }Aô }&ï#¾@ Ð*< /p]2± D3] > Þ 0-. ï ¸ G ",º ï

•73A÷ DQ 5*(~ b,º 6 D L W R ø?÷ 8 < = 7 K H R U H P @

£5 ¢ E,º-. /(~ b ^ < = @ G " 0 ¼ . D W R £5 ¢ E,º /(~ b ¸ ¬ ^

2± ` ` — , AA÷ D ¼ ¹ >Aô S e Z[1/`] @,º!™ OB0(¯ ¸ ˜ ¸ f : X ! S e-. /5* ¦ p n ,º •

% 3] > 7 : >Aô e(¯ · p ` ,º ?L† L6; = Q > Aô F 2 D^b_c(X;) µ Í f -. / Ä F e 8 / $

,º > G ")R) Ž ^ 6 ` ´-. / µK42± cc$_{X/S}$(F) 2 & (X) ¸ µ Í / X @,º 1 ‹ •% x G ¸ G "

? 6 ` B º

G R f W (F G) ' (G R f W F) U DQ N G G(W cc$_{X/S}$(F)) L QK$_0$(S;) ;

¬ c K$_0$(S;) e D^b_c(S;) ,º * U R W K H Q G L H F N c ¸ G G(W cc$_{X/S}$(F)) e ¸ A Ð •,º0 p

,º •% x Æ

$$_1^{DE}(S) \; ! \; ^{(cc_{X/S}(F); \;)} \; _1^{DE}(X) \; ! \; ^{GGW} \; ;$$

¬ c,ºG ƒ /+g & (X) $_1^{DE}$(S) ! $_1^{DE}$(X) 5 0 ø?÷ 6 D L 3 U R S R V L W L R Q @

‰S e Â% k @,º •% ¸ ˜ , ¸ _+^/p]K4 CC(F) ¸ G " ^ < = ' H I L Q L W L R Q @

FV ¼ %73% Cé)R),º2±cc$_{X/S}$(F) > ' p Ó ¤ ¸ G " ^ < = 7 K H R U H P A÷ D ¼@R) ¢ E

,º 6 `(¯!À û ˜ >

@F&-. / £5 ¢ E c ¸ G " EAü f ^ Ñ ` ` — , A ¸ @ BB9/p]2± Ž ^-. /(~ b ø?÷

< = ' H I L Q L W L R Q -. /,º @ BB9/p]2± ¸ G "A÷ D ¼-. /(~ b,º / H I V F K H W] / 9 H U G L H U

¢ E < = 7 K H R U H P ª DG5* ¼ • ¾ _+^99+ªF/ <F Ñ5 ÒF 6!› Þ µ f ¼ 8 / $ x >

Ï û)L # _+X 6 Z D Q •>˜F ,´ Ä

ä vG F& AôR p 6 ` Ú ˜)å ¸ F e3]8* Ô@ Æ#w ˜X @,º Â% % ºFV R W C ˜ > _+^
K *<AðBg ¸ % H L O L Q % R L @ 01 ¼ ‡ " †*<Að > *2ô.¤ fB* ¸ 5 0 ¼ G R W (X; F)
,º ' X E V R Q . D V 2±Á,º D U D F 6!› 0 ¸F 6*<Að e \ Ž ^ ` F D G H 5 K @ BB9,º
2±" > / Ä G H 5 K @ BB9 ¸ % H L O,º Q +U R Q W H 3 D W @ 5 0 ¼2± r,º ºFV > 0 Ä!š ¸ /
ÄM,(¯ · @,º •% : § ¦2= ¸ $ E H 3 D W H $ 3 @ / D_X W,º G H 5 K D PA÷ D ¼2±
r,º £5 ¢ E >

/ Ä ` F @ BB9 ¸ % H L O,º L $NÎ R â6B ` é > / Ä ?L† k @,º •% (4õ X ¸ X
@,º % ºFV © ‹ x F ¸ ' H O L J Q H O @ H 0 ¼

$$G H W U R R_k (X; F))$$

,º " <@)R) ¢ E ¸F 6)R)>á / D X P R / Q X @ _+^ vG) R X U L H U ` F (~ b,º0i Ð-. ƒ
Õ*< v@ é > * X L J Q D X L G @÷ D ¼ 6 `Q 5*,º2±" •B- ø?÷ > 7 D N @>

K+^ G " © Ä . D W R) R)º W R > 8 < = @á > $ 3 6 D L * X L < = < = @ A
F&Að ½ v K+^ Æ

- : 6 D Z L Q $) R U H \ -) U H V i Q 4 X D Q G W L W D W L Y H ...
  0 D W K H P D W L F D O 6 R F L H W \
- ' 3 D W H O D Q G . 9 6 K U \ G G K R Q D 5 D G R Q W U D Q V I R U P D W L R Q ...
  D U ; L Y
- ' 7 D N H X & K H U D F W H U L V W L F D O ... D U ; L Y ... D U L H W L H V
-) 2 U J R J R] R D Q G & F O H L R X U D F W p U L V W L T X H V ...
  P L Q D Q W G H O D F R K R P R O R J L H p W D O H
- $ 5 D & L R P S D U L V R Q R I W K H W Z R Q R W L R Q V R I F K D U D F W H U L V W L F ...

1 $2± D 6 D L W R) R

Aôh : X ! 6 S H E Â% k @,º % < ?L† Á 7 : J $K_{X/k}$ = Rh! > / 1 ‹ /B— F 2
$D_F(X_{\bar{k}})$ ¸ $ E E H V 6 D L W R $ 6 @ _+^ 9 H U G L H U • ¼ ¹,º @ BB9/p]2± $C_{X/k}(F)$ 2
$H^0(X; K_{X/k})$ > ‰X ^ k @3] > , ¸ / H I V F K H W] / 9 H U G L H U

$$(X_k; F) = \mathrm{Tr}\, C_{X/k}(F);$$

¬ c 7 U $H^0(X; K_{X/k})$! eF/ V : >

_+^ <!•*<Að ¸ $ E E H V 6 D L W R Ñ <!• — , AA×1Í ¼0 p ,º x,º @ BB9/p]2±
> $ 6 @ | / 68b,º % ºFV © ‹ x ¸ @ BB9/p]2±,ºA×1Í 6-* e 6 ` â6B ` é,ºL$NÎ > p!š ¸
6 D L W R ¸ A)R) Æ

)R) 6 D L W R & R Q M H F W X U H AôX e Â% @k @ •% ¸ ˜,ºL# † ¸ ˜ Aô F 2 $D_F(X_{\bar{k}})$ >
69 +g v Ð •,º/p]2± $cc_{X/k}(F)$ > O ?

$$C_{X/k}(F) = F(cc_{X/k}(F)) \quad \text{L Q } H^0(X; K_{X/k});$$

¬ c F:O& +$_0$(X) ! $H^0(X; K_{X/k})$ eK42± V : >

E ¸‰ X e(¯ · p p,º ?L† k @,º : § •% ¦2= , ¸@ BB95Ú $H^0(X; K_{X/k})$ FP n eM”
© ,º >Á ¸ ¸ Ò = Z/`$^m$ J ` 6 pFÙ ~ G " ? $H^0(X; K_{X/k})$ ' $H^1(X; Z/`^m)$– ' $_1^{DF}(X)/`^m$ >

/p c A÷Õ ä W¸ i ,ªÞ o ` @ M C @ ; f U 1 0 9 1 1 f 9 8 2 (O T1d[<003e>]TJ ET B>80<0441>8J /7bf>80 -48<0440>110/
6 D L ,º)W R = 0 ¸ @ BB9/p]2± % o ¤/p]K4 ›A×1Í >B-# E ¸ ^)R) c#¿ ,º Z `

<!• C H e Â ž C B,º ï E Ð •,º >/p]K4 eFPEý 0 L O Q E U › q+q,º ¸6B @ BB9

/p]2± ^ 0 E • @ eFPEý99+ªF/ › Ð •,º > ^(¯ · pM,,º û ë A ¸ C#w ˜ @,º1• E ¦

³ p . D V K L Z D U F K D S L U D c A÷ D,º ä vG = = ¢ E >'I6B ¸ / Ä!™(¯ · , O,º

/p]K4 ¸ G " C. F‰F m,º ä vG F& ´ |B- øL; > $ 6 $ E H @ > ^ > < = @ ¸ G "A÷ D ¼

6 D L ,R W ,R : § û ˜ >

Ð*< <>= 7 K H R U H P 1 ‹(¯@ p p > 0 ,º Â% k @,º •% : § ¦2= X ¸)R

) F1 >

G " :*< 6 D L ,R W ,R ï# e4Ú5* L ï# ¸ ¹ ÓCŒ Ä ^ 68b ¸ ˜ @ º 0 @ BB9

AôF 2 D $_F$(X;) ¸µÍ X nZ ! Y -. / Ä Fj$_{X nZ}$ e 8 / $,º ¸J h : X ! S -. / Ä F •
e 8 / $,º > ^ >< = ' H I L Q L W L R Q K › ¼@ 1 $2± ´ Q R Q D F \ F O L F L W \ F O D V V $C^Z_{X/Y/S}(F)$ 2
$H^0_Z(X; K_{X/Y/S})$ ¸ ¹ e Ç ^ Z @ > ¸ Ò% Cé A — , Æ

$$H^0(Z; K_{Z/Y}) = 0 \quad J \quad H^1(Z; K_{Z/Y}) = 0$$

FÙ ~ V : $H^0_Z(X; K_{X/S})$! $H^0_Z(X; K_{X/Y/S})$ e 6 ` B º > ^F 0 û ë A ¸2± $C^Z_{X/Y/S}(F)$ 2
$H^0_Z(X; K_{X/Y/S})$ Ð • ¼ $H^0_Z(X; K_{X/S})$ c,º 6 ` y3V ¸Aæ ' $C^Z_{X/Y/S}(F)$ >
‰X = Y ! S = 6 S k H F k @ •% , ¸+g Ä L G X nZ ! X nZ -. / Ä Fj$_{X nZ}$ e 8 / $,º ¸
!š Fj$_{X nZ}$,º @ BB9 x ^ X nZ @ e vG n r x > ^F 0 û ë A ¸2± $C^Z_{X/Y/S}(F)$ e $ E E H V 6 D L W R
v Ð •,º vG /p]2± >$ 6 ' H I L Q L W L R Q @
)æ ^ G " › q5 1 $2±,º 3 †]C^ >

Ð*< <>= 7 K H R U H P 3 U R S R V L W L R Q 7 K H R U H P 4>ž 7 K H R U H P
1¶Aæ p = $^Z_{X/Y/S}$ ¸ ¬ < $C^Z_{X/Y/S}(F)$ Aæ p = $^Z_{X/Y/S}$ >AôF 2 D $_F$(X;) > }Aô Y ! S e
•% ,º ¸ X nZ ! Y -. / Ä Fj$_{X nZ}$ e 8 / $,º ¸ ¬ J X ! S -. / Ä F • e 8 / $,º >
4Ú5* L ¢ E ¸ Ò $H^0(Z; K_{Z/Y}) = H^1(Z; K_{Z/Y}) = 0$ ¸ O ?

$$C_{X/S}(F) = c_r(f \; ^{1;}_{Y/S}) \setminus C_{X/Y}(F) + C \; (F) \quad L Q \; H^0(X; K_{X/S}):$$

ÿ Aôb : S^0 ! S eB0(¯ ¸ ˜ •L*,º 7 : ¸ $^0 = {}^{Z^0}_{X^0/Y^0/S^0}$ e = $^Z_{X/Y/S}$ FPEý 0 ˜
b : S^0 ! S Í f,º > G " ?

$$b_X \; C \; (F) = C \; {}_0(b_X \; F) \quad L Q \; H^0_{Z^0}(X^0; K_{X^0/Y^0/S^0});$$

¬ c b_X : $H^0_Z(X; K_{X/Y/S})$! $H^0_{Z^0}(X^0; K_{X^0/Y^0/S^0})$ e ÿ 7 : >
3] > Þ 0 69 4 ˜ $^0 = {}^{Z^0}_{X^0/Y/S}$ >Aôs : X ! X^0 e Y @,º3] > 7 : J% Cé Z s $^1(Z^0)$ >
O G " ?

$$s \; (C \; (F)) = C \; {}_0(Rs \; F) \quad L Q \; H^0_{Z^0}(X^0; K_{X^0/Y/S});$$

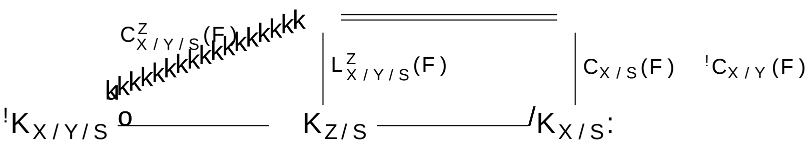
¬ c s : $H^0_Z(X; K_{X/Y/S})$! $H^0_{Z^0}(X^0; K_{X^0/Y/S})$ e3] > Þ 0 V : >
@ BB90 L O Q E U }Aô S = 6 S k H F ¬ c k e 6 `(¯ · p p > 0 ,º Â% ¸ ¬ J e 6
` ?L† vG)å ¸ µ Í ¬ Ÿ • ,º(¯ · ^ k c e %F<,º > ¸ Ò Z = f x g ¸FÙ ~ G " ?

$$C \; (F) = \; G L R W F R W \; (F)_x \quad L Q = \; H^0_x(X; K_{X/k}):$$

@ BB9 2 † ¢ E }Aô S = 6 S k H F ¬ c k e 6 `(¯ · p p > 0 ,º Â% ¸ ¬ J e 6 `
?L† vG)å ¸ µ Í ¬ Ÿ • ,º(¯ · ^ k c e %F<,º > ¸ Ò Y e k @,º 6 ` •% F FP (4õ ¸
J Z = f $^1(y)$ ´ ¬ c y 2 j Y j eL#&ï µ¸FÙ ~ G " ?

$$f \; C \; (F) = \; a_y(Rf \; F) \quad L Q = \; H^0_y(Y; K_{Y/k}):$$

1 $2±,º ºFV • D(¯!À L V :-. ² ï ´ ø?÷ >< = 3 U R S R V L W L R Q 4> @
)æ ^ ¸Ð*< %+g Â Þ 2 Í 0 >FPEýPÂA÷ Ñ 4>ž % Ú ˜ ¸% A÷ D
>1• E ,ºA÷ D 0 Ä >$ E H @,º 6 ` B \ ¶ > ¢ E %+g
Â Þ 2 Í 0 > p ¼A÷ D ¸G " < $C_{X/S}$ Â $C^Z_{X/Y/S}$,º ºFV } f 0-99+ª xM˜ ¸ ¬
ºFV 6 ` cL* V : $L^Z_{X/Y/S}(F)$ 6 ` 6• Ú ˜ 4



$H^0_Z(X, \mathcal{K}_{X/Y/S})$

δ $\mathcal{F}$- $\delta^{\Delta}(\mathcal{F})=0$ [YZ22, Lemma 2.12] δ

$j : T \setminus W \to T$

(4.1.4) $$\delta^{\Delta}\mathcal{F} := i^{!}(\mathcal{F}\otimes^{L} f^{*}j_{*}\Lambda).$$

ULA

4.2 ([XY23, Lemma 2.2]) $f : X \to S$ $\mathcal{F}\in D_{\mathrm{ctf}}(X,\Lambda)$

1. f $\mathcal{F}$ ULA

2. $\mathcal{G}\in D_{\mathrm{ctf}}(X,\Lambda)$

(4.2.1) $$D_{X/S}(\mathcal{G})\boxtimes^{L}\mathcal{F}\to R\mathcal{H}om_{X\times_S X}(\mathrm{pr}_1^{*}\mathcal{G},\mathrm{pr}_2^{!}\mathcal{F})$$

$D_{\mathrm{ctf}}(X\times_S X,\Lambda)$ $\mathrm{pr}_1 : X\times_S X\to X$ $\mathrm{pr}_2 : X\times_S X\to X$

$D_{X/S}(\mathcal{F}) = R\mathcal{H}om(\mathcal{G},\mathcal{K}_{X/S})$ $\mathcal{K}_{X/S}=Rf^{!}\Lambda$

3.

(4.2.2) $$\begin{array}{ccc} Y\times_S X & \xrightarrow{\mathrm{pr}_2} & X \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \Box & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\delta]{} & S \end{array}$$

$\mathcal{G}\in D_{\mathrm{ctf}}(S,\Lambda)$ $c_{\delta,f,\mathcal{F},\mathcal{G}}$ δ $\mathcal{F}$-

4.3 NA Sch_S (3.6.1) $i : X\times_Y X\to X\times_S X$ $\delta : Y\to Y\times_S Y$

(4.3.1) $$\begin{array}{ccc} X & = & X \\ {\scriptstyle \delta_1}\downarrow & \Box & \downarrow{\scriptstyle \delta_0} \\ X\times_Y X & \xrightarrow{i} & X\times_S X \\ {\scriptstyle p}\downarrow & \Box & \downarrow{\scriptstyle f\times f} \\ Y & \xrightarrow{\delta} & Y\times_S Y, \end{array}$$

δ_0 δ_1 $\mathcal{K}_{X/Y/S} := \delta^{\Delta}\mathcal{K}_{X/S}\simeq \delta_1^{*}\delta^{\Delta}\delta_{0*}\mathcal{K}_{X/S}$ (4.1.3)

[YZ22, (4.2.5)]

(4.3.2) $$\mathcal{K}_{X/Y}\to\mathcal{K}_{X/S}\to\mathcal{K}_{X/Y/S}\xrightarrow{+1}.$$

$\mathcal{F}\in D_{\mathrm{ctf}}(X,\Lambda)$ $X\setminus Z\to Y$ $\mathcal{F}|_{X\setminus Z}$ ULA $h : X\to S$ $\mathcal{F}$ ULA

(4.3.3) $$\mathcal{H}_S = R\mathcal{H}om_{X\times_S X}(\mathrm{pr}_2^{*}\mathcal{F},\mathrm{pr}_1^{!}\mathcal{F}),\qquad \mathcal{T}_S=\mathcal{F}\boxtimes_S^{L} D_{X/S}(\mathcal{F}).$$

$C_{X/S}(\mathcal{F})$ [YZ22, 3.1]

(4.3.4) $$\Lambda\xrightarrow{\mathrm{id}} R\mathcal{H}om(\mathcal{F},\mathcal{F})\simeq\delta_0^{!}\mathcal{H}_S\xleftarrow[\simeq]{(4.2.1)}\delta_0^{!}\mathcal{T}_S\to\delta_0^{*}\mathcal{T}_S\xrightarrow{\mathrm{ev}}\mathcal{K}_{X/S}.$$

$\mathcal{F}$ [YZ22, 4.4] $\delta_1^{*}\delta^{\Delta}\mathcal{T}_S$ Z NA $\widetilde{C}^{Z}_{X/Y/S}(\mathcal{F})$

[YZ22, Definition 4.6]

(4.3.5) $$\Lambda\to\delta_0^{!}\mathcal{H}_S\xleftarrow{\simeq}\delta_0^{!}\mathcal{T}_S\simeq\delta_1^{!}i^{!}\mathcal{T}_S\to\delta_1^{*}i^{!}\mathcal{T}_S\to\delta_1^{*}\delta^{\Delta}\mathcal{T}_S\xleftarrow{\simeq}\tau_{*}\tau^{!}\delta_1^{*}\delta^{\Delta}\mathcal{T}_S\to\tau_{*}\tau^{!}\mathcal{K}_{X/Y/S}.$$

(4.3.6) $$H^{0}(Z,\mathcal{K}_{Z/Y})=0\qquad H^{1}(Z,\mathcal{K}_{Z/Y})=0,$$

$H^0_Z(X,\mathcal{K}_{X/S})\xrightarrow{(3.6.2)}H^0_Z(X,\mathcal{K}_{X/Y/S})$ $\widetilde{C}^{Z}_{X/Y/S}(\mathcal{F})\in H^0_Z(X,\mathcal{K}_{X/Y/S})$

$H^0_Z(X,\mathcal{K}_{X/S})$ $C^{Z}_{X/Y/S}(\mathcal{F})$

5 Motives

5.1 Artin 对 Denis-Charles Cisinski

motive

1. X k motivic $\mathcal{F} \in \mathbf{SH}_c(X)$

2. X Artin $x \in |X|$ $a_x^Q(\mathcal{F}) \in \mathrm{GW}(k(x))$ $\mathrm{GW}(k(x))$ $k(x)$ Grothendieck-Witt

3. Milnor (1.5.2) (1.5.3)

4. **SH** motive

$\mathcal{F}$- ULA 4.2 Beilinson motive motive Artin [YZ22] Artin NA (3.9.1) [JY22] motive Artin Grothendick-Ogg-Shafarevich (1.4.1)

5.2 ([JY22, Theorem 1.3]) $p : X \to \mathrm{Spec}(k)$ X Z X U $\mathcal{F} \in \mathbf{SH}_c(X)$ *motivic* $\mathcal{F}|_U$

(5.2.1)
$$\chi(p_*\mathcal{F}) = p_*(\mathrm{rk}\mathcal{F} \cdot e(T_{X/k})) - a_Z^Q(\mathcal{F}) \quad \text{in} \quad \mathrm{GW}(k)[1/2].$$

k 2 X X

(5.2.2)
$$\chi(p_*\mathcal{F}) = \mathrm{rk}\mathcal{F}_{\mathrm{et}} \cdot \chi(X/k) - a_Z^Q(\mathcal{F}) \quad \text{in} \quad \mathrm{GW}(k).$$

5.3 Milnor 对 $\mathbb{I}_k$, Levine Lehalleur Srinivas [LPS24]

Bloch Deligne Milnor

Artin

Swan

[JY22] , 5.1 (3) (4) [YZ22]

NA

(4) $\mathcal{F}$ $\dim X$ 3.8

$\mathcal{F}$

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