RESEARCH STATEMENT
Characteristic classes and ″-factors for constructible etale sheaves

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Contents

Overview. My research focuses on geometric rami cation theory for constructible etale sheaves and its motivic counterpart. In the past ve years (2019-2023), I have published three peer-reviewed articles [UYZ20, YZ21, JY21] and four preprints [YZ22, JY22, JSY22, XY23] with my collaborators. The main achievements of these articles are as follows:

(1) In [UYZ20], we prove a twist formula for the ″-factor of a constructible sheaf, which is conjectured[1] by Kato and Saito in [KS08, Conjecture 4.3.11].
(2) In [YZ21], we propose a relative version of Kato-Saito's twist formula. As an evidence of this conjecture, we generalize the cohomological characteristic class de ned by Abbes and Saito to a relative case under certain transversality conditions. This notion is further generalized to universal local acyclicity (ULA) sheaves by Lu and Zheng in [LZ22].
(3) In [YZ22], we construct a cohomological characteristic class (called non-acyclicity class) supported on the non-acyclicity locus. Using this class, we con rm the quasi-projective case of Saito's conjecture [Sai17], namely that the cohomological characteristic classes de ned by Abbes and Saito can be computed in terms of the characteristic cycles. As other applications, we prove cohomological analogs of the Milnor formula and the conductor formula for constructible sheaves on (not necessarily smooth) varieties.
(4) In [XY23], we de ne the geometric counterpart of the non-acyclicity class and formulate a Milnor-type formula for non-isolated singularities, which says that the non-acyclicity classes can be calculated in terms of the characteristic cycles.
(5) In the papers [JY21, JY22, JSY22], we study rami cation theory for motives. We propose a quadratic version of the Artin conductor for SH motives and then construct a quadratic version of the Grothendieck-Ogg-Shafarevich formula.

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[1] The original conjecture uses Swan class, but in our paper, we replace it with characteristic class.

This research statement is organized as follows. In Section 1, we give a quick overview of geometric ramification theory. Section 2 introduces our work on Kato-Saito's conjecture on the twist formula of ε-factors. In Section 3, we summarize the properties of non-acyclicity classes and discuss Saito's conjecture on characteristic classes. In Section 4, we recall the construction of non-acyclicity classes. In Section 5, we present a review of our work on ramification theory for motives.

1. Ramification theory

In this section, we provide a brief overview of ramification theory, concentrating specifically on the discussion of characteristic classes and characteristic cycles for constructible etale sheaves due to personal constraints.

1.1. Let S be a Noetherian scheme and Sch_S the category of separated schemes of finite type over S. Let Λ be a finite local ring such that the characteristic of its residue field is invertible on S. For any scheme $X \in \mathrm{Sch}_S$, we denote by $D_{\mathrm{ctf}}(X;\Lambda)$ the derived category of complexes of Λ-modules of finite tor-dimension with constructible cohomology groups on X.

1.2. Consider the following assumptions on S:

- (G) S is the spectrum of a perfect field k of characteristic $p > 0$.
  In this geometric case, we have a well-defined number $\mathrm{dim}K = \mathrm{rank}K$ for $K \in D_{\mathrm{ctf}}(S;\Lambda)$.
- (A) S is the spectrum of a discrete valuation ring. Let η be the generic point of S and s its closed point. We assume that the residue field $k(s)$ is a perfect field of characteristic $p > 0$.
  In this arithmetic case, for any $K \in D_{\mathrm{ctf}}(S;\Lambda)$, we have the Swan conductor $\mathrm{Sw}K$ (measuring the wild ramification), the total dimension $\mathrm{dimtot}K = \mathrm{dim}K_{\bar\eta} + \mathrm{Sw}K$ and the Artin conductor $a_S(K) = \mathrm{dim}K_{\bar\eta} - \mathrm{dim}K_{\bar s} + \mathrm{Sw}K = \mathrm{dimtot}R\Phi_{\mathrm{id}}(K)$, where $R\Phi$ is the vanishing cycles functor.

In ramification theory, there exist three distinct versions of higher-dimensional analogues of Artin/Swan conductors: the cohomological characteristic class introduced by Abbes and Saito in [AS07], the Swan class presented by Kato and Saito in [KS08, KS12], and the characteristic class/cycle constructed by Saito in [Sai17] based on Beilinson's singular support. These classes are related to the following Riemann-Roch type questions:

Question 1.3. Let $f : X \to S$ be a separated morphism of finite type and $F \in D_{\mathrm{ctf}}(X;\Lambda)$.

- In the geometric case (G), how to compute $\chi_c(X_{\bar k}; F) = \mathrm{dim}Rf_!F$?
- In the arithmetic case (A), how to compute $\mathrm{Sw}Rf_!F$, $\mathrm{dimtot}Rf_!F$ and $a_S(Rf_!F)$?

These problems are previously studied by Abbes [Abb00], Abbes-Saito [AS07], Bloch [Blo87], Deligne [Del72, Del11], Hu [Hu15], Kato-Saito [KS04, KS08, KS12], Laumon [Lau83], Saito [Sai17, Sai18, Sai21] and Tsushima [Tsu11].

Based on the observation that the Swan conductor can be defined through the logarithmic localized intersection product, Kato and Saito [KS08, KS12] explore ramification theory using logarithmic geometry and K-theoretic localized intersection theory. Their methodology give rise to the so-called Swan class, which can be regarded as a higher-dimensional generalization of the Swan conductor. Recently, Abe [Abe21] introduces a homotopical/∞-categorical way to study ramification theory.

In [YZ22], we use a cohomological way to study Question 1.3 by introducing a cohomological class (called non-acyclicity class) supported on the non-acyclicity locus Z (Z is the smallest closed subset of X

1.4. Grothendieck-Ogg-Shafarevich formula. Consider the geometric case (G). Assume X is a proper smooth connected curve over $S = \mathrm{Spec}\,k$. Then the Euler-Poincare characteristic $\chi(X_{\bar{k}}, \mathcal{F})$ is computed by the Grothendieck-Ogg-Shafarevich (GOS) formula

$$\chi(X_{\bar{k}}, \mathcal{F}) = \mathrm{dim}\mathcal{F}_{\bar{\xi}} \cdot \chi(X_{\bar{k}}, \Lambda) - \sum_{x \in Z} a_x(\mathcal{F}) \cdot \deg(x), \tag{1.4.1}$$

where ξ is the generic point of X, Z is a finite set of closed points such that $\mathcal{F}|_{X \setminus Z}$ is smooth and $a_x(\mathcal{F}) = a_{X_{(x)}}(\mathcal{F})$ is the Artin conductor of $\mathcal{F}$ at x. By the Gauss-Bonnet-Chern formula $\chi(X_{\bar{k}}, \Lambda) = \deg(c_1(\Omega^{1,\vee}_{X/k}) \cap [X])$, the formula (1.4.1) can be rewritten as follows

$$\chi(X_{\bar{k}}, \mathcal{F}) = \deg(cc_{X/k}(\mathcal{F})), \tag{1.4.2}$$

where $cc_{X/k}(\mathcal{F})$ is a zero-cycle class on X:

$$cc_{X/k}(\mathcal{F}) = \mathrm{dim}\mathcal{F}_{\bar{\xi}} \cdot c_1(\Omega^{1,\vee}_{X/k}) \cap [X] - \sum_{x \in Z} a_x(\mathcal{F}) \cdot [x] \quad \text{in} \quad \mathrm{CH}_0(X). \tag{1.4.3}$$

1.5. Characteristic cycle. There is a generalization of the GOS formula to higher dimensional case by using characteristic cycles.[2] In the transcendental setting [KS90], Kashiwara and Schapira give a microlocal description of the characteristic cycle for a constructible sheaf $\mathcal{F}$ on a manifold without using D-modules. In [Bei07], Beilinson asks if there is a motivic (ℓ-adic or de Rham) counterpart for their theory. As observed by Deligne, there is a strong analogy between the wild ramification of etale constructible sheaves in positive characteristic and the irregular singularity of partial differential equations on a complex manifold. In [Del11], Deligne proposes a general program to define characteristic cycles of constructible etale sheaves. Deligne's program is achieved by Saito [Sai17, Theorem 4.9 and Theorem 6.13] based on the singular support defined by Beilinson [Bei16].

Let X be a connected smooth variety of dimension n over k. For any $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$, the singular support $SS(\mathcal{F})$ is the smallest closed conical subset of the cotangent bundle T^*X such that locally on X, every function $f : X \to \mathbb{A}^1_k$ with df disjoint from $SS(\mathcal{F})$ is locally acyclic relatively to $\mathcal{F}$. It is proved in [Bei16] that $SS(\mathcal{F})$ is of dimension n. Later, Saito [Sai17] constructs an n-cycle $CC(\mathcal{F})$ supported on $SS(\mathcal{F})$ with $\mathbb{Z}$-coefficients, which satisfies the following properties

(a) (Index formula) Assume that X is projective[3] over a perfect field k. Then we have

$$\chi(X_{\bar{k}}, \mathcal{F}) = (CC(\mathcal{F}), T^*_X X)_{T^*X}, \tag{1.5.1}$$

where $T^*_X X$ is the zero section of T^*X.

(b) (Milnor formula) Let $f : X \to C$ be a flat morphism from a smooth scheme X to a smooth curve C over k. Assume that x_0 is an isolated characteristic point of f with respect to $SS(\mathcal{F})$ (cf. [Sai17, Definition 3.7]). Then

$$-\mathrm{dimtot}\, R\Phi_f(\mathcal{F})_{\overline{x}_0} = (CC(\mathcal{F}), df)_{T^*X, x_0}. \tag{1.5.2}$$

(c) (Conductor formula) Let X be a smooth scheme over k and Y a smooth connected curve over k with the generic point η. Let $\mathcal{F} \in D^b_c(X, \Lambda)$. Let $f : X \to Y$ be a quasi-projective morphism such that f is proper on the support of $\mathcal{F}$ and is properly $SS(\mathcal{F})$-transversal over an open dense sub-scheme $V \subseteq Y$. For each closed point $y \in Y$, the Artin conductor $a_y(Rf_*\mathcal{F}) = \chi(X_{\overline{\eta}}, \mathcal{F}) - \chi(X_{\overline{y}}, \mathcal{F}) + \mathrm{Sw}_y R\Gamma(X_{\overline{\eta}}, \mathcal{F})$ is computed by the following (geometric) conductor formula

$$-a_y(Rf_*\mathcal{F}) = (CC(\mathcal{F}), df)_{T^*X, y}. \tag{1.5.3}$$

[2] Kato and Saito [KS08, KS12] obtain another higher dimensional GOS formula and its arithmetic version by introducing Swan classes for constructible étale sheaves. See 2.4 for more details.

[3] Abe obtains an index formula for proper varieties by using ∞-categories in [Abe21].

When F is the constant etale sheaf , then $CC(\ ) = (-1)^n \cdot [T^*_X X]$ and (1.5.1) is the Gauss-Bonnet-Chern formula $(X_{\bar{k}},\ ) = \deg(c_n(\ ^{1,\vee}_{X/k}) \cap [X])$. The formula (1.5.2) is equivalent to the following classical Milnor formula (cf. [Del73, Conjecture 1.9, P200])

$$-\mathrm{dimtot}\, R\ \ _f(\ )_{\bar{x}_0} = (-1)^n \cdot \mathrm{length}$$

is the epsilon factor (the constant term in the functional equation (2.1.1)) and $\chi(X_{\bar{k}}, \mathcal{F})$ is the Euler-Poincare characteristic of $\mathcal{F}$. In the functional equation (2.1.1), both $\chi(X_{\bar{k}}, \mathcal{F})$ and $\varepsilon(X, \mathcal{F})$ are related to the ramification theory. Indeed, $\chi(X_{\bar{k}}, \mathcal{F}) = \deg cc_X(\mathcal{F})$ (cf. (1.5.1) and (1.7.1)). For the epsilon factor, it is more complicated.

2.2. **Twist formula.** Let $\rho_X : CH_0(X) \to \pi_1(X)^{\mathrm{ab}}$ be the reciprocity map which is defined by sending the class $[s]$ of a closed point $s \in X$ to the geometric Frobenius Frob_s. Let $\mathcal{G}$ be a smooth sheaf on X and $\det \mathcal{G} : \pi_1(X)^{\mathrm{ab}} \to \Lambda^\times$ be the character associated to the determinant sheaf $\det \mathcal{G}$. In joint work with Umezaki and Zhao [UYZ20], we prove the following twist formula:

$$\varepsilon(X, \mathcal{F} \otimes \mathcal{G}) = \det \mathcal{G}(-cc_X(\mathcal{F})) \cdot \varepsilon(X, \mathcal{F})^{\mathrm{rank}\mathcal{G}}, \tag{2.2.1}$$

which is conjectured by Kato and Saito in [KS08, Conjecture 4.3.11].[4] When $\mathcal{F}$ is the constant sheaf Λ, this is proved in [Sai84]. If $\mathcal{F}$ is a smooth sheaf on an open dense subscheme U of X such that the complement $D = X \backslash U$ is a simple normal crossing divisor and the sheaf $\mathcal{F}$ is tamely ramified along D, then (2.2.1) is a consequence of [Sai93, Theorem 1]. If $\dim(X) = 1$, the formula (2.2.1) follows from the product formula of Deligne and Laumon (cf. [Del72e, 7.11] and [Lau87, 3.2.1.1]). In [Vid09a, Vid09b], Vidal proves a similar result on a proper smooth surface over a finite field of characteristic $p > 2$ under some technical assumptions.

As a corollary of (2.2.1), we prove the compatibility of the characteristic class with proper push-forward by using the injectivity of the reciprocity map ρ_X [KS83, Theorem 1]. In general, Saito proves that the characteristic cycle (resp. characteristic class) is compatible with proper push-forward under a mild assumption (cf. [Sai17, 7.2], [Sai18, Conjecture 1] and [Sai21, Theorem 2.2.5]). In [YZ21], we also prove a relative version of the twist formula (2.5.1).

Question 2.3. Prove a similar formula of (2.2.1) if $\mathcal{G}$ is smooth only on an open dense subscheme $U \subseteq X$ such that its wild ramification along $X \backslash U$ is much smaller than that of $\mathcal{F}$.

2.4. **Swan class.** To generalize the classical Grothendieck-Ogg-Shafarevich formula for curves to higher dimensional varieties, Kato and Saito define the so-called Swan class in [KS08]. Saito formulates a conjecture that this object should be re-defined using the characteristic cycle (cf. [Sai17, Conjecture 5.8]). More precisely, let X be a smooth scheme over a perfect field k of characteristic $p > 0$, and $\overline{X}$ a smooth compactification of X. For a smooth and constructible sheaf $\mathcal{F}$ of Λ-modules on X, Saito conjectures that the Swan class of $\mathcal{F}$ should have integer coefficients and is equal to the pull back by the zero section of the difference $CC(j_!\mathcal{F}) - \mathrm{rank}\mathcal{F} \cdot CC(j_!\Lambda)$. In [UYZ20], we verify a weaker version of this conjecture for smooth surfaces over a finite field. Our method also works for higher dimensional varieties if we assume resolution of singularities and a special case of proper push-forward of characteristic class (cf. [UYZ20, Theorem 6.6]).

2.5. **Relative twist formula.** In [YZ21, 2.1], we formulate a relative version of Kato-Saito's formula and prove it under certain transversality conditions. Let S be a regular Noetherian scheme over $\mathbb{Z}[1/\ell]$ and $f : X \to S$ a smooth proper morphism purely of relative dimension n. Let Λ be a finite field of characteristic ℓ or $\Lambda = \overline{\mathbb{Q}}_\ell$. Let $\mathcal{F} \in D^b_c(X, \Lambda)$ such that f is universally locally acyclic relatively to $\mathcal{F}$. We conjecture that there is a (relative) cycle class $cc_{X/S}(\mathcal{F}) \in \mathrm{CH}^n(X)$ such that for any smooth sheaf $\mathcal{G}$ of Λ-modules on X, we have an isomorphism

$$\det Rf_*(\mathcal{F} \otimes^L \mathcal{G}) \simeq (\det Rf_*\mathcal{F})^{\otimes \mathrm{rank}\mathcal{G}} \otimes^L \det\mathcal{G}(cc_{X/S}(\mathcal{F})) \quad \text{in} \quad K_0(S, \Lambda), \tag{2.5.1}$$

where $K_0(S, \Lambda)$ is the Grothendieck group of $D^b_c(S, \Lambda)$. In (2.5.1), the object $\det\mathcal{G}(cc_{X/S}(\mathcal{F}))$ is a smooth sheaf of rank 1 determined as follows:

$$\pi_1^{\mathrm{ab}}(S) \xrightarrow{(cc_{X/S}(\mathcal{F}), -)} \pi_1^{\mathrm{ab}}(X) \xrightarrow{\det\mathcal{G}} \Lambda^\times, \tag{2.5.2}$$

[4] The original conjecture is formulated in terms of the Swan class.

where the pairing is given by $\mathrm{CH}^n(X) \times \pi_1^{\mathrm{ab}}(S) \to \pi_1^{\mathrm{ab}}(X)$ (cf. [Sai94, Proposition 1]).

When S is a smooth scheme over a perfect field k, we construct a candidate for $cc_{X/S}(F)$ in [YZ21, Definition 2.11] by using the characteristic cycle of $CC(F)$. As an evidence, we prove a special case of the conjectural formula (2.5.1) in [YZ21, Theorem 2.12].

From the above relative twist formula, we realize that there is a relative version of the cohomological characteristic class (cf. [YZ21, Definition 3.6]) under certain transversality conditions. We also prove a relative Lefschetz-Verdier trace formula in [YZ21, Theorem 3.9]. These results are further generalized to ULA sheaves by Lu and Zheng [LZ22] by using categorical traces.

2.6. **Microlocal description.** Let R be a commutative ring. Let F be a perfect constructible complex of sheaves of R-modules on a compact real analytic manifold X. In [Bei07], Beilinson develops the theory of topological epsilon factors using K-theory spectrum. More precisely, he gives a Dubson-Kashiwara-style description of $\det R\Gamma(X,F)$, and he asks that whether the construction admits a motivic (ℓ-adic or de Rham) counterpart. For de Rham cohomology, such a construction is given by his PhD student Patel in [Pat12]. Based on these, Abe and Patel prove a similar twist formula in [AP18] for global de Rham epsilon factors in the classical setting of D_X-modules on smooth projective varieties over a field of characteristic zero. As pointed out by Abe and Patel, proving the formula at the level of K-theory spectra should also give formulas in higher K-theory. At the level of K_0 (resp. K_1), one gets formulas for the Euler characteristic (resp. determinants). It would be interesting to see the consequences at the level of K_2 (or higher K-groups).

For ℓ-adic cohomology, Beilinson's question is still open. For a constructible etale sheaf F on a smooth curve X over a finite field k, the precise statement for the ε-factorization of

$$\det(-\mathrm{Frob}_k; R\Gamma(X,F))$$

was conjectured by Deligne [Del72e] and proved by Laumon [Lau87] using local Fourier transform and ℓ-adic version of principle of stationary phase. A higher dimensional analogue is obtained by Guignard [Gui22] (see also [Tak19]).

2.7. **Citation.** Our work [UYZ20] on Kato-Saito's conjecture is cited by [AP18, Sai21, Gui22, YZ21, YZ22] and also by the following papers:

(1) W. Sawin, A. Forey, J. Fresan and E. Kowalski, *Quantitative sheaf theory*, Journal of the American Mathematical Society, 36(3), (2023): 653-726.
(2) D. Patel and K. V. Shuddhodan, *Brylinski-Radon transformation in characteristic $p>0$*, preprint arXiv:2307.04156, 2023.
(3) D. Takeuchi, *Characteristic epsilon cycles of ℓ-adic sheaves on varieties*, arXiv:1911.02269, 2019.
(4) F. Orgogozo and J. Riou, *Cycle caracteristique sur une puissance symetrique d'une courbe et determinant de la cohomologie etale*, arXiv:2312.07776, 2023.
(5) A. Rai, *Comparison of the two notions of characteristic cycles*, arXiv:2312.09945, 2023.

3. Non-acyclicity class and Saito's conjecture

3.1. Let $h: X \to \mathrm{Spec}k$ be a separated morphism of finite type over a perfect field k. Let $K_{X/k} = Rh^!\Lambda$. For any object $F \in D_{\mathrm{ctf}}(X,\Lambda)$, the cohomological characteristic class $C_{X/k}(F) \in H^0(X, K_{X/k})$ is introduced by Abbes and Saito in [AS07] by using Verdier pairing. If X is proper over k, the Lefschetz-Verdier trace formula gives

$$\chi(X_{\bar{k}}, F) = \mathrm{Tr}C_{X/k}(F), \tag{3.1.1}$$

where $\mathrm{Tr}: H^0(X, K_{X/k}) \to \Lambda$ is the trace map.

Using ramification theory, Abbes and Saito calculate the cohomological characteristic classes for rank 1 sheaves under certain ramification conditions in [AS07]. However, the calculation for general constructible etale sheaves remains an outstanding question in ramification theory. In general, Saito proposes the following conjecture.

Conjecture 3.2 (Saito, [Sai17, Conjecture 6.8.1]). *Let X be a closed sub-scheme of a smooth scheme over a perfect field k. Let F be a constructible complex of Λ-modules of finite tor-dimension on X. Consider the characteristic class $cc_{X/k}(F)$ defined by (1.7.1). Then we have*

$$C_{X/k}(F) = \mathrm{cl}(cc_{X/k}(F)) \quad \text{in} \quad H^0(X, K_{X/k}), \tag{3.2.1}$$

where $\mathrm{cl} : \mathrm{CH}_0(X) \to H^0(X, K_{X/k})$ *is the cycle class map.*

Note that when X is projective and smooth over a finite field k of characteristic p, the cohomology group $H^0(X, K_{X/k})$ is highly non-trivial. For example, if $\Lambda = \mathbb{Z}/\ell^m$ with $\ell \neq p$, then we have $H^0(X, K_{X/k}) \simeq H^1(X, \mathbb{Z}/\ell^m)^\vee \simeq \pi_1^{\mathrm{ab}}(X)/\ell^m$.

Saito's conjecture says that the cohomological characteristic class can be computed in terms of the characteristic cycle. Note that the two involved ramification invariants in Conjecture 3.2 are defined in quite different ways. The characteristic cycle is characterized by the Milnor formula (1.5.2), while the cohomological characteristic class in some sense is defined via the categorical trace. In the characteristic zero case, the equality (3.2.1) on a complex manifold is the microlocal index formula proved by Kashiwara and Schapira [KS90, 9.5.1]. However, we don't know such a microlocal description for characteristic cycles in positive characteristic (but see [AS09, Abe21]). In [YZ22], we prove the quasi-projective case of Saito's conjecture.

Theorem 3.3 ([YZ22, Theorem 1.3]). *Conjecture 3.2 holds for any smooth and quasi-projective scheme X over a perfect field k of characteristic $p > 0$.*

3.4. Our approach to Saito's conjecture is the fibration method, which leans on the construction of the relative version of cohomological characteristic classes over a general base scheme. To achieve this, it is necessary to impose additional transversality conditions on the structure morphism. Let S be a Noetherian scheme. Let $h : X \to S$ be a separated morphism of finite type, $K_{X/S} = Rh^!\Lambda$ and $F \in D_{\mathrm{ctf}}(X, \Lambda)$. In fact, under certain smooth and transversality conditions on h, we introduce the relative (cohomological) characteristic class $C_{X/S}(F) \in H^0(X, K_{X/S})$ in [YZ21, Definition 3.6]. It is further generalized to any separated morphism $h : X \to S$

prove Saito's conjecture by induction on the dimension of X, and the curve case follows from the Grothendieck-Ogg-Shafarevich formula (1.4.3) and its cohomological version (curve case of (3.5.1)).

3.6. In order to prove Theorem 3.5, we have to give a purely cohomological/categorical way to de ne the right hand side of (3.5.1), i.e., we have to de ne a cohomological class supported on the non-acyclicity locus. Let S be a Noetherian scheme. Consider a commutative diagram in Sch_S:

(3.6.1)
$$\begin{array}{ccccc} Z & \overset{\tau}{\hookrightarrow} & X & \xrightarrow{f} & Y; \\ & & & h\searrow \quad \swarrow g & \\ & & & S & \end{array}$$

where $: Z \to X$ is a closed immersion and g is a smooth morphism. We de ne an object $K_{X/Y/S}$ on X sitting in a distinguished triangle (see also [YZ22, (4.2.5)])

(3.6.2)
$$K_{X/Y} \to K_{X/S} \to K_{X/Y/S} \xrightarrow{+1} :$$

Let $F \in D_{\mathrm{ctf}}(X; \ )$ such that $X \backslash Z \to Y$ is universally locally acyclic relatively to $F|_{X\backslash Z}$ and that $h : X \to S$ is universally locally acyclic relatively to F. In [YZ22, De nition 4.6], we introduce the non-acyclicity class $\widetilde{C}^Z_{X/Y/S}(F) \in H^0_Z(X; K_{X/Y/S})$ supported on Z. If the following condition holds:

(3.6.3)
$$H^0(Z; K_{Z/Y}) = 0 \text{ and } H^1(Z; K_{Z/Y}) = 0$$

then the map $H^0_Z(X; K_{X/S}) \xrightarrow{(3.6.2)} H^0_Z(X; K_{X/Y/S})$ is an isomorphism. In this case, the class $\widetilde{C}^Z_{X/Y/S}(F) \in H^0_Z(X; K_{X/Y/S})$ de nes an element of $H^0_Z(X; K_{X/S})$, which is denoted by $C^Z_{X/Y/S}(F)$.

In the case that $X = Y \to S = \mathrm{Spec}\,k$ is smooth over a eld k. Since $\mathrm{id} : X\backslash Z \to X\backslash Z$ is universally locally acyclic relatively to $F|_{X\backslash Z}$, the cohomology sheaves of $F|_{X\backslash Z}$ are locally constant on $X\backslash Z$. In this case, the class $C^Z_{X/Y/S}(F)$ is Abbes-Saito's localized characteristic class [AS07, De nition 5.2.1].

Now we summarize the functorial properties for the non-acyclicity classes.

Theorem 3.7 ([YZ22, Theorem 1.9, Proposition 1.11, Theorem 1.12, Theorem 1.14]). *Let us denote*

2;). d[(In)K87(aF60 ...

(4) (*Cohomological Milnor formula*) *Assume* $S = \operatorname{Spec} k$ *for a perfect field* k *of characteristic* $p > 0$ *and* Λ *is a finite local ring such that the characteristic of the residue field is invertible in* k. *If* $Z = \{x\}$, *then we have*

$$C_{\Delta}(\mathcal{F}) = -\mathrm{dimtot} R\Phi_f(\mathcal{F})_{\bar{x}} \quad \text{in} \quad \Lambda = H^0_x(X, \mathcal{K}_{X/k}). \tag{3.7.4}$$

(5) (*Cohomological conductor formula*) *Assume* $S = \operatorname{Spec} k$ *for a perfect field* k *of characteristic* $p > 0$ *and* Λ *is a finite local ring such that the characteristic of the residue field is invertible in* k. *If* Y *is a smooth connected curve over* k *and* $Z = f^{-1}(y)$ *for a closed point* $y \in |Y|$, *then we have*

$$f_*C_{\Delta}(\mathcal{F}) = -a_y(Rf_*\mathcal{F}) \quad \text{in} \quad \Lambda = H^0_y(Y, \mathcal{K}_{Y/k}). \tag{3.7.5}$$

The formation of non-acyclicity classes is also compatible with specialization maps (cf. [YZ22, Proposition 4.17]).

Now Theorem (3.5) follows from (3.7.1) and (3.7.4). By verifying certain diagrams commute, one could prove (3.7.1)-(3.7.3). The proof of (3.7.4) is based on (3.7.2) together with a homotopy argument in [Abe22]. The formula (3.7.5) follows from (3.7.3) and (3.7.4). To prove (3.7.1), we enhance the constructions of $C_{X/S}$ and $C^Z_{X/Y/S}$ to the ∞-categorical level and construct an intermediate map $L^Z_{X/Y/S}(\mathcal{F})$ together with a coherent commutative diagram

$$\begin{array}{ccccc} & & \Lambda & = & \Lambda \\ & \swarrow^{C^Z_{X/Y/S}(\mathcal{F})} & \downarrow{\scriptstyle L^Z_{X/Y/S}(\mathcal{F})} & & \downarrow{\scriptstyle C_{X/S}(\mathcal{F})-\delta^! C_{X/Y}(\mathcal{F})} \\ i_*i^!\mathcal{K}_{X/Y/S} & \longleftarrow & i_*\mathcal{K}_{Z/S} & \longrightarrow & \mathcal{K}_{X/S}. \end{array} \tag{3.7.6}$$

Since $H^0_Z(X, \mathcal{K}_{X/Y/S}) \simeq H^0(Z, \mathcal{K}_{Z/S})$, the diagram (3.7.6) implies the fibration formula (3.7.1).

Remark 3.8. If we apply the non-acyclicity class to the following diagram constructed by Saito in [Sai17, p.652, (5.13)]

$$\begin{array}{ccccc} \mathbf{Z}(\widetilde{\mathbf{C}}) & \hookrightarrow & (X \times \mathbf{G})^{\triangledown} & \longrightarrow & \mathbf{D}, \\ & & & \searrow \quad \swarrow & \\ & & & \mathbf{G} & \end{array} \tag{3.8.1}$$

we could be able to recover the characteristic cycle $CC(\mathcal{F})$ in a weaker sense.

3.9. **Cohomological expression for Artin conductors.** Let X be a smooth connected curve over k. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ and $Z \subseteq X$ be a finite set of closed points such that the cohomology sheaves of $\mathcal{F}|_{X \setminus Z}$ are locally constant. By the cohomological Milnor formula (3.7.4), we have the following (motivic) expression for the Artin conductor of $\mathcal{F}$ at $x \in Z$

$$a_x(\mathcal{F}) = \mathrm{dimtot} R\Phi_{\bar{x}}(\mathcal{F}, \mathrm{id}) = -C^{\{x\}}_{U/U/k}(\mathcal{F}|_U), \tag{3.9.1}$$

where U is any open subscheme of X such that $U \cap Z = \{x\}$. By (3.7.1), we get the following cohomological Grothendieck-Ogg-Shafarevich formula (cf. [YZ22, Corollary 6.6]):

$$C_{X/k}(\mathcal{F}) = \mathrm{rank}\mathcal{F} \cdot c_1(\Omega^{1,\vee}_{X/k}) - \sum_{x \in Z} a_x(\mathcal{F}) \cdot [x] \quad \text{in} \quad H^0(X, \mathcal{K}_{X/k}). \tag{3.9.2}$$

Based on the observation (3.9.1), we could be able to study the ramification theory for motives and get a quadratic version of the GOS formula (cf. [JY22]).

3.10. **Milnor-type formula for non-isolated singularities.** In [XY23], we construct the geometric counterpart of the non-acyclicity class and propose a Milnor-type formula for non-isolated singularities. The conjecture says that the non-acyclicity classes can be calculated in terms of the characteristic cycles.

4. Transversality condition

In this section, we recall the de nition of the non-acyclicity class. To simplify our notation, we omit to write R or L to denote the derived functors unless otherwise stated explicitly or for $RHom$.

4.1. **Transversality condition.** We recall the (cohomological) transversality condition introduced in [YZ22, 2.1], which is a relative version of the transversality condition studied by Saito [Sai17, De nition 8.5]. Let S be a Noetherian scheme and a Noetherian ring such that $m = 0$ for some integer m invertible on S. Consider the following cartesian diagram in Sch_S:

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ W & \xrightarrow{\delta} & T. \end{array} \tag{4.1.1}$$

Let $F \in D_{\mathrm{ctf}}(Y,)$ and $G \in D_{\mathrm{ctf}}(T,)$. Let $c_{\delta,f,\mathcal{F},\mathcal{G}}$ be the composition

$$\begin{aligned} c_{\delta,f,\mathcal{F},\mathcal{G}} : i^*F \otimes^L p^* \ ^!G &\xrightarrow{id\otimes \mathrm{b.c}} i^*F \otimes^L i^! f^*G \xrightarrow{\mathrm{adj}} i^! i_!(i^*F \otimes^L i^! f^*G) \\ &\xrightarrow[\simeq]{\mathrm{proj.formula}} i^!(F \otimes^L i_! i^! f^*G) \xrightarrow{\mathrm{adj}} i^!(F \otimes^L f^*G). \end{aligned} \tag{4.1.2}$$

We put $c_{\delta,f,\mathcal{F}} := c_{\delta,f,\mathcal{F},\Lambda} : i^*F \otimes^L p^* \ ^! \ \to i^!F$. If $c_{\delta,f,\mathcal{F}}$ is an isomorphism, then we say that the morphism is *F-transversal*. If $c_{i,\mathrm{id},\mathcal{F}}$ is an isomorphism, then we say i is F-transversal.

By [YZ22, 2.11], there is a functor $^{\Delta} : D_{\mathrm{ctf}}(Y,) \to D_{\mathrm{ctf}}(X,)$ such that for any $F \in D_{\mathrm{ctf}}(Y,)$, we have a distinguished triangle

$$i^*F \otimes^L p^* \ ^! \ \xrightarrow{c_{\delta,f,\mathcal{F}}} i^!F \to \ ^{\Delta}F \xrightarrow{+1} . \tag{4.1.3}$$

Then is F-transversal if and only if $^{\Delta}(F)=0$ (cf. [YZ22, Lemma 2.12]). If is a closed immersion and $j : T\backslash W \to T$ is the open immersion, then we have

$$^{\Delta}F := i^!(F \otimes^L f^* j_* \ ). \tag{4.1.4}$$

The following lemma gives an equivalence characterization between transversality condition and (universal) local acyclicity condition (cf. [XY23, Lemma 2.2]).

Lemma 4.2. *Let $f : X \to S$ be a morphism of nite type between Noetherian schemes and $F \in D_{\mathrm{ctf}}(X,)$. The following conditions are equivalent:*

(1) *The morphism f is (universally) locally acyclic relatively to F.*
(2) *For any $G \in D_{\mathrm{ctf}}(X,)$, the canonical map*

$$D_{X/S}(G) \,\mathrm{b}^L F \to RHom_{X\times_S X}(\mathrm{pr}_1^* G, \mathrm{pr}_2^! F) \tag{4.2.1}$$

is an isomorphism in $D_{\mathrm{ctf}}(X \times_S X,)$, where $\mathrm{pr}_1 : X \times_S X \to X$ and $\mathrm{pr}_2 : X \times_S X \to X$ are the projections, $D_{X/S}(F) = RHom(G, K_{X/S})$ and $K_{X/S} = Rf^! \ $.

(3) *For any cartesian diagram between Noetherian schemes*

$$\begin{array}{ccc} Y\times_S X & \xrightarrow{\mathrm{pr}_2} & X \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \square & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\delta]{} & S \end{array} \tag{4.2.2}$$

and any $G\in D_{\mathrm{ctf}}(S,\Lambda)$*, the morphism* $c_{\delta,f,\mathcal{F},\mathcal{G}}$ *is an isomorphism (in particular,* δ *is* F*-transversal).*

4.3. **Non-acyclicity class.** Consider the commutative diagram (3.6.1). Let $i: X\times_Y X\to X\times_S X$ be the base change of the diagonal morphism $\delta: Y\to Y\times_S Y$:

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle \delta_1}\downarrow & \square & \downarrow{\scriptstyle \delta_0} \\ X\times_Y X & \xrightarrow{i} & X\times_S X \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle f\times f} \\ Y & \xrightarrow{\delta} & Y\times_S Y, \end{array} \qquad (f: X\to Y) \tag{4.3.1}$$

where δ_0 and δ_1 are the diagonal morphisms. Put $K_{X/Y/S}:=\delta^\Delta K_{X/S}\simeq\delta_1^*\delta^\Delta\delta_{0*}K_{X/S}$. By (4.1.3), we have the following distinguished triangle (cf. [YZ22, (4.2.5)])

$$K_{X/Y}\to K_{X/S}\to K_{X/Y/S}\xrightarrow{+1}. \tag{4.3.2}$$

Let $F\in D_{\mathrm{ctf}}(X,\Lambda)$ such that $X\backslash Z\to Y$ is universally locally acyclic relatively to $F|_{X\backslash Z}$ and that $h: X\to S$ is universally locally acyclic relatively to F. We put

$$H_S=RHom_{X\times_S X}(\mathrm{pr}_2^*F,\mathrm{pr}_1^!F),\qquad T_S=F\boxtimes_S^L D_{X/S}(F). \tag{4.3.3}$$

The relative cohomological characteristic class $C_{X/S}(F)$ is the composition (cf. [YZ22, 3.1])

$$\Lambda\xrightarrow{\mathrm{id}}RHom(F,F)\simeq\delta_0^!H_S\xleftarrow[\simeq]{(4.2.1)}\delta_0^!T_S\to\delta_0^*T_S\xrightarrow{\mathrm{ev}}K_{X/S}. \tag{4.3.4}$$

By the assumption on F, $\delta_1^*\delta^\Delta T_S$ is supported on Z by [YZ22, 4.4]. The non-acyclicity class $\widetilde{C}^Z_{X/Y/S}(F)$ is the composition (cf. [YZ22, Definition 4.6])

$$\Lambda\to\delta_0^!H_S\xleftarrow{\simeq}\delta_0^!T_S\simeq\delta_1^!i^!T_S\to\delta_1^*i^!T_S\to\delta_1^*\delta^\Delta T_S\xleftarrow{\simeq}\iota_*\iota^!\delta_1^*\delta^\Delta T_S\to\iota_*\iota^!K_{X/Y/S}. \tag{4.3.5}$$

If the following condition holds:

$$H^0(Z,K_{Z/Y})=0\text{ and }H^1(Z,K_{Z/Y})=0, \tag{4.3.6}$$

then the map $H^0_Z(X,K_{X/S})\xrightarrow{(3.6.2)}H^0_Z(X,K_{X/Y/S})$ is an isomorphism. In this case, the class $\widetilde{C}^Z_{X/Y/S}(F)\in H^0_Z(X,K_{X/Y/S})$ defines an element of $H^0_Z(X,K_{X/S})$, which is denoted by $C^Z_{X/Y/S}(F)$.

5. Ramification theory for motives

5.1. **Quadratic Artin conductor.** When I was doing postdoc at Regensburg University, Professor Denis-Charles Cisinski proposed a project on constructing the characteristic cycles for motives. To carry out this project, we have to consider the following things:

(1) Define the singular support for a constructible motivic spectrum $F\in\mathrm{SH}_c(X)$ on a smooth variety X over a perfect field k.

(2) Construct a quadratic re nement of the Artin conductor in the case that X is a smooth curve. More precisely, for each closed point $x \in |X|$, we need a quadratic form $a_x^Q(F)$ in the Grothendieck-Witt ring $\mathrm{GW}(k(x))$ of (virtual) non-degenerate symmetric bi-linear forms over $k(x)$ such that the rank of $a_x^Q(F)$ equals the classical Artin conductor at x of the etale realization of F.
(3) Formulate a quadratic re nement for the Milnor formula (1.5.2) and the conductor formula (1.5.3).
(4) Construct a quadratic version of the characteristic cycle for a nice SH motive.

In an ongoing note with Cisinski, we could be able to de ne the singular support for constructible motives following Beilinson's argument by using F-transversality conditions instead of the universal local acyclicity conditions (cf. Lemma 4.2). However at that time, it is di cult to de ne the Artin conductor for a motive. Later in [YZ22], we observe that the Artin conductor for an etale constructible sheaf can be expressed in terms of the non-acyclicity class (cf. (3.9.1)). In the joint work [JY22] with Fangzhou Jin, we have successfully de ned the Artin conductor of a constructible motive and formulate a quadratic version of the Grothendick-Ogg-Shafarevich formula (1.4.1).

Theorem 5.2 ([JY22, Theorem 1.3]). *Let $p : X \to \mathrm{Spec}(k)$ be a smooth and proper morphism with X connected, and let Z be a nowhere dense closed subscheme of X with open complementary U. Let $F \in$ SH*

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