



## 1. Introduction

EM (ML) (MAP) 3. EM ( . . , 2,4,12,15,16,18,19 ). G , EM . , A 9 , EM 5 , -N ” 7 . 6,10 , H , EM M 13 17 , 7 8 , - ( . . , ). I , EM J 20 H EM G H EM , EM EM , , EM . M , EM . -N EM ML MAP . M . 14 I EM G . B EM J 20 , EM . , EM I , 18 , EM 14 G , EM . F , 14 G .

$$\begin{aligned}
 & \text{EM} \\
 & \text{I} \\
 & \text{G} \quad \text{EM} \\
 & \quad \quad \quad \text{4} \quad \quad \quad \text{5.}
 \end{aligned}$$

## 2. The EM algorithm for mixtures of densities from exponential families

### 2.1. The mixture model

$$\begin{aligned}
 & : \\
 P(x|F) &= \sum_{i=1}^K a_i P_i(x|f_i), \quad a_i \geq 0, \quad \sum_{i=1}^K a_i = 1, \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 & x = [x_1, \dots, x_n] \in R^n, \quad P_i \\
 & \quad \quad \quad f_i \in O_i \subset R^{d_i}, \quad K \\
 & \quad \quad \quad F \\
 & \quad \quad \quad f_i, \quad F = (a_1, \dots, a_K, f_1, \dots, f_K) \in O, \quad a_i \\
 O &= \left\{ (a_1, \dots, a_K, f_1, \dots, f_K) : \sum_{i=1}^K a_i = 1, \quad a_i \geq 0, f_i \in O_i \quad i = 1, \dots, K \right\}.
 \end{aligned}$$

$$\begin{aligned}
 & I \quad P_i(x|f_i) = P_i(x|m_i, S_i) \quad G \\
 P_i(x|f_i) &= P(x|m_i, S_i) = \frac{1}{(2\pi)^{n/2} (S_i)^{1/2}} \exp\left\{ -\frac{1}{2}(x-m_i)^T S_i^{-1} (x-m_i) \right\}, \quad (2)
 \end{aligned}$$

$$\begin{aligned}
 & m_i = [m_{i1}, \dots, m_{in}] \quad G, \quad S_i = (s_{kl}^i)_{n \times n} \\
 & \quad \quad \quad I, \quad EM \\
 & G \\
 & 14. I, \quad EM
 \end{aligned}$$

$$\begin{aligned}
 & G \\
 A \quad & q(x|y), y \in Y \subset R^d \quad R^n
 \end{aligned}$$

$$\begin{aligned}
 q(x|y) &= a(y)^{-1} b(x)^{y \cdot t(x)}, \quad x \in R^n, \quad (3) \\
 & b(x), t(x) \quad x \quad R^n \quad a(y)
 \end{aligned}$$

$$a(y) = \int_{R^n} b(x)^{y \cdot t(x)} \, m$$

$$\begin{aligned}
 & m \quad R^n. I \quad b(x) \geq 0 \\
 x \in R^n, a(y) < +\infty \quad y \in Y \quad t(x), \quad , \\
 & A, \quad b(x).
 \end{aligned}$$



## 2.2. The EM algorithm and its asymptotic convergence rate

$$\begin{aligned} \mathcal{S}_N &= \{x^{(t)} : t = 1, \dots, N\} \\ \mathbf{f}_1, \dots, \mathbf{f}_K & \text{ ML} \\ \mathbf{F} &= (\mathbf{a}_1, \dots, \mathbf{a}_K, \mathbf{f}_1, \dots, \mathbf{f}_K) \\ L(\mathbf{F}) &= \sum_{t=1}^N P(x^{(t)}|\mathbf{F}) \\ \text{E} & \text{ (1), EM} \end{aligned}$$

$$\mathbf{a}_i^+ = \frac{1}{N} \sum_{t=1}^N \frac{\mathbf{a}_i P_i(x^{(t)}|\mathbf{f}_i)}{P(x^{(t)}|\mathbf{F})}, \quad (7)$$

$$\mathbf{f}_i^+ = \left\{ \sum_{t=1}^N t_i(x^{(t)}) \frac{\mathbf{a}_i P_i(x^{(t)}|\mathbf{f}_i)}{P(x^{(t)}|\mathbf{F})} \right\} / \left\{ \sum_{t=1}^N \frac{\mathbf{a}_i P_i(x^{(t)}|\mathbf{f}_i)}{P(x^{(t)}|\mathbf{F})} \right\}, \quad (8)$$

$$i = 1, \dots, K.$$

$$\begin{aligned} L(\mathbf{F}) & \text{ 3.19 . M} \\ \text{EM} & \\ L(\mathbf{F}) & \text{ 18 . I} \\ \mathbf{F}^* & \\ \mathcal{S}_N & \\ \text{EM} & \\ \mathbf{F}^N & \\ \mathbf{F}^* & \end{aligned}$$

$$\mathbf{F}^+ = G(\mathbf{F}) \quad \text{EM}$$

$$\mathbf{F}^+ - \mathbf{F}^N = G(\mathbf{F}) - G(\mathbf{F}^N) = G'(\mathbf{F}^N)(\mathbf{F} - \mathbf{F}^N) + O(\|\mathbf{F} - \mathbf{F}^N\|^2) \quad (9)$$

$$\mathbf{F} - \mathbf{F}^N, \quad G'(\mathbf{F}) \quad \mathbf{J} \quad G(\mathbf{F}) - \mathbf{F}^N \quad O(x)$$

$$E(G'(\mathbf{F}^*)) = I - Q(\mathbf{F}^*)R(\mathbf{F}^*), \quad G'(\mathbf{F}^N)$$

$$Q(\mathbf{F}^*) = \text{diag}(\mathbf{a}_1^*, \dots, \mathbf{a}_K^*, \mathbf{a}_1^{*-1} \mathbf{P}_1, \dots, \mathbf{a}_K^{*-1} \mathbf{P}_K) \quad (10)$$

$$\mathbf{P}_i = \int_{R^n} [t_i(x) - \mathbf{f}_i^*][t_i(x) - \mathbf{f}_i^*] P_i(x|\mathbf{f}_i^*) \, \mathbf{m}$$

$$R(\mathbf{F}^*) = \int_{R^n} V(x)V(x) P(x|\mathbf{F}^*) \, \mathbf{m} \quad (11)$$

$$V(x) = (\mathbf{b}_1(x), \dots, \mathbf{b}_K(x), \mathbf{a}_1^* \mathbf{b}_1(x) \mathbf{G}_1(x), \dots, \mathbf{a}_K^* \mathbf{b}_K(x) \mathbf{G}_K(x)) ,$$

$$\mathbf{b}_i(x) = P_i(x|\mathbf{f}_i^*)/P(x|\mathbf{F}^*),$$

$$\mathbf{G}_i(x) = \mathbf{P}_i^{-1}[t_i(x) - \mathbf{f}_i^*].$$

$$\begin{aligned}
& \text{H} \quad \quad \quad , E(\cdot) = E_{\mathbf{F}^*}(\cdot). \text{ I} \quad \quad \quad \mathbf{F}^N \quad \quad \quad \text{E} \quad . \quad (9) \\
& \quad \quad \quad \text{EM} \quad \quad \quad N \quad \quad \quad , \quad \quad \quad \text{EM} \quad \quad \quad \mathbf{F}^*: \\
& \quad \quad \quad r \leq \lim_{N \rightarrow \infty} \|G'(\mathbf{F}^N)\| = \left\| \lim_{N \rightarrow \infty} G'(\mathbf{F}^N) \right\| \\
& \quad \quad \quad = \|E(G'(\mathbf{F}^*))\| = \|I - Q(\mathbf{F}^*)R(\mathbf{F}^*)\|. \quad (12) \\
& \text{I} \quad \quad \quad , \quad \quad \quad \text{EM}
\end{aligned}$$

### 3. The main result

#### 3.1. The measures of the overlap

$$\begin{aligned}
& \text{G} \quad \quad \quad . \quad \quad \quad \mathbf{F}^*: \quad \quad \quad \text{E} \quad . \\
& (1) \quad \quad \quad \mathbf{F}^*:
\end{aligned}$$

$$h_i(x) = \frac{\mathbf{a}_i^* P_i(x|\mathbf{f}_i^*)}{\sum_{j=1}^K \mathbf{a}_j^* P_j(x|\mathbf{f}_j^*)} \quad i = 1, \dots, K. \quad (13)$$

$$\begin{aligned}
& \text{I} \quad \quad \quad \text{E} \quad . \quad (11) \\
& h_i(x) = \mathbf{a}_i^* \mathbf{b}_i(x). \quad (14)
\end{aligned}$$

$$\begin{aligned}
& \mathbf{g}_{ij}(x) = (\mathbf{d}_{ij} - h_i(x))h_j(x) \quad i, j = 1, \dots, K, \quad (15) \\
& \mathbf{d}_{ij} \quad \quad \quad \mathbf{K} \quad \quad \quad . \quad \quad \quad , \\
& \quad \quad \quad \vdots
\end{aligned}$$

$$e_{ij}(\mathbf{F}^*) = \int_{R^n} |\mathbf{g}_{ij}(x)| P(x|\mathbf{F}^*) \, \mathbf{m}$$

$$\begin{aligned}
& i, j = 1, 2, \dots, K, \quad e_{ij}(\mathbf{F}^*) \leq 1 \quad |\mathbf{g}_{ij}(x)| \leq 1. \\
& \text{F} \quad i \neq j, e_{ij}(\mathbf{F}^*) \\
& \quad \quad \quad i \quad j \\
& \quad \quad \quad x, h_i(x)h_j(x) \quad . \quad P_i(x|\mathbf{f}_i^*) \quad P_j(x|\mathbf{f}_j^*)
\end{aligned}$$

$$\begin{aligned}
 e(F^*) &= 0 \\
 h_i(x)h_j(x) &= 0 \quad i \neq j \\
 I &= \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \quad \text{EM} \quad N \\
 &\quad \cdot H \quad , \\
 &\quad \cdot I \quad , \\
 &\quad e(F^*) \quad \cdot A \\
 &\quad \text{EM} \\
 &\quad e(F^*) \quad .
 \end{aligned}
 \tag{18}$$

### 3.2. Regular conditions and lemmas

(1) *Nondegenerate condition on the mixing proportions:*

$$\begin{aligned}
 a_i^* &\geq a \quad i = 1, \dots, K, \\
 a &= \min_{i=1, \dots, K} a_i^* .
 \end{aligned}
 \tag{16}$$

(2) *Uniform attenuating condition on the eigenvalues of the covariance matrices:*

$$\begin{aligned}
 S_i^* &= \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \quad i = 1, \dots, K \\
 \lambda_1(F^*) &\leq \lambda_{ij} \leq \lambda_n(F^*) \quad i = 1, \dots, K, \quad k = 1, \dots, n, \\
 \lambda_1(F^*) &= \min_{i,j} \lambda_{ij} \\
 S_1^*, \dots, S_K^* &= \begin{pmatrix} 1(F^*) \\ \vdots \\ 1(F^*) \end{pmatrix} \\
 \lambda_1(F^*) &= \min_{i,j} \lambda_{ij} \\
 B &= \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \\
 E &= \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \quad K
 \end{aligned}
 \tag{17}$$

$$1 \leq k(S_i^*) \leq B' \quad i = 1, \dots, K,$$

$$k(S_i^*) \leq B' \quad S_i^* \leq B' .$$

(3) *Regular condition on the mean vectors:*

$$\begin{aligned}
 & \quad \quad \quad , \dots, m_1^*, \dots, m_K^*, \\
 & \mathbf{m} D \quad (\mathbf{F}^*) \leq D \quad (\mathbf{F}^*) \leq \|m_i^* - m_j^*\| \leq D \quad (\mathbf{F}^*) \quad i \neq j, \quad (18) \\
 & D \quad (\mathbf{F}^*) = \quad_{i \neq j} \|m_i^* - m_j^*\|, \quad D \quad (\mathbf{F}^*) = \quad_{i \neq j} \|m_i^* - m_j^*\|, \quad \mathbf{m} \\
 & \quad \quad \quad , \\
 & \quad \quad \quad . \mathbf{M} \quad , \\
 & \quad \quad \quad , \quad \quad \quad m_i^*, m_j^* \\
 & \quad \quad \quad T \quad \quad \quad \|m_i^* - m_j^*\| \geq T \quad i \neq j. \text{ I} \quad \quad \quad , \quad \quad \quad , \\
 & \text{E} . (18) \quad \quad \quad m_1^*, \dots, m_K^* \\
 & \quad \quad \quad \|P_i^{-1}\|, \quad \quad P_i \quad \quad \quad \text{E} . (10). \\
 & \quad \quad \quad P_i(x|\mathbf{f}_i^*) \quad \quad \quad ,
 \end{aligned}$$

11 :

$$P_i =$$



$$\begin{aligned}
& Z(F^*) \rightarrow 0, \quad Z(F^*) \rightarrow 0, \quad e(F^*) \rightarrow 0, \quad Z(F^*) \rightarrow 0 \\
& A, \quad Z(F^*) \rightarrow 0, \quad e(F^*) \rightarrow 0, \quad F^* \\
& Z(F^*), \quad f(Z) = \frac{e(F^*)}{Z(F^*)=Z} \quad (21)
\end{aligned}$$

$$\begin{aligned}
& e(F^*) \quad 1. B, \\
& e_{ij}(F^*) \leq e(F^*) \leq f(Z(F^*)) \quad i \neq j. \quad (22) \\
& F, \quad (A \quad A).
\end{aligned}$$

**Lemma 1.** Suppose that a mixture of  $K$  densities from the bell sheltered exponential families of the parameter  $F^*$  satisfies Conditions (1)–(3). As  $Z(F^*)$  tends to zero, we have

- ( )  $Z(F^*), Z_i(m_i^*)$  and  $Z_j(m_j^*)$  are the equivalent infinitesimals.  
 ( ) For  $i \neq j$ , we have

$$\|m_i^*\| \leq T' \|m_i^* - m_j^*\|, \quad (23)$$

where  $T'$  is a positive number.

- ( ) For any two nonnegative numbers with  $p + q > 0$ , we have

$$\|m_i^* - m_j^*\|^p (1^i)^{-nq} \leq O(Z^{-p \vee q}(F^*)), \quad (24)$$

where  $p \vee q = \{p, q\}$ .

**Lemma 2.** Suppose that a mixture of  $K$  densities from the bell sheltered exponential families of the parameter  $F^*$  satisfies Conditions (1)–(3). As  $Z(F^*)$  tends to zero, we have for each  $i$

$$\|P_i\| \leq c \|m_i^* - m_j^*\|^p, \quad (25)$$

where  $j \neq i$ ,  $c$  and  $p$  are some positive numbers.

$$E(\|t_i(X) - f_i^*\|^2) \leq u M_i^q(F^*), \quad (26)$$

where  $M_i(F^*) = \sum_{j \neq i} \|m_i^* - m_j^*\|$ ,  $u$  and  $q$  are some positive numbers.

**Lemma 3.** Suppose that a mixture of  $K$  densities from the bell sheltered exponential families of the parameter  $F^*$  satisfies Conditions (1)–(3) and  $Z(F^*) \rightarrow 0$  as an infinitesimal, we have

$$f^e(Z(F^*)) = o(Z^p(F^*)), \quad (27)$$

where  $\epsilon > 0$ ,  $p$  is any positive number and  $o(x)$  means that it is a higher order infinitesimal as  $x \rightarrow 0$ .

$$e(F^*) \leq \frac{Z(F^*)}{L} \leq \frac{3}{L} E(F^*),$$

### 3.3. The main theorem

**Theorem 1.** Given a mixture of  $K$  densities from the bell sheltered exponential families of the parameter  $F^*$  that satisfies Conditions (1)–(4), as  $e(F^*)$  tends to zero as an infinitesimal, we have

$$r \leq \|E(G'(F^*))\| = o(e^{0.5-\epsilon}(F^*)), \quad (28)$$

where  $\epsilon$  is an arbitrarily small positive number.

A mixture of  $K$  densities from the bell sheltered exponential families of the parameter  $F^*$  that satisfies Conditions (1)–(4), as  $e(F^*)$  tends to zero as an infinitesimal, we have

Let  $r = \|E(G'(F^*))\|$ , then by (28), we have

$$r \leq o(e^{0.5-\epsilon}(F^*)) = o(e^{0.5-\epsilon}(F^*)^{1/2}) = o(e^{0.5-\epsilon}(F^*)^{1/2})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/4})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/8})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/16})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/32})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/64})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/128})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/256})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/512})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1024})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2048})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/4096})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/8192})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/16384})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/32768})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/65536})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/131072})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/262144})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/524288})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1048576})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2097152})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/4194304})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/8388608})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/16777216})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/33554432})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/67108864})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/134217728})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/268435456})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/536870912})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1073741824})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2147483648})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/4294967296})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/8589934592})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/17179869184})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/34359738368})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/68719476736})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/137438953472})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/274877906944})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/549755813888})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1099511627776})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2199023255552})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/4398046511104})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/8796093022208})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/17592186044416})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/35184372088832})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/70368744177664})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/140737488355328})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/281474976710656})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/562949953421312})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1125899906842624})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2251799813685248})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/4503599627370496})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/9007199254740992})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/18014398509481984})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/36028797018963968})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/72057594037927936})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/144115188075855872})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/288230376151711744})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/576460752303423488})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1152921504606846976})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2305843009213693952})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/4611686018427387904})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/9223372036854775808})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/18446744073709551616})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/36893488147419103232})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/73786976294838206464})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/147573952589676412928})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/295147905179352825856})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/590295810358705651712})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1180591620717411303424})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2361183241434822606848})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/4722366482869645213696})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/9444732965739290427392})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/18889465931478580854784})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/37778931862957161709568})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/75557863725914323419136})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/151115727451828646838272})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/302231454903657293676544})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/604462909807314587353088})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1208925819614629174706176})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2417851639229258349412352})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/4835703278458516698824704})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/9671406556917033397649408})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/19342813113834066795298816})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/38685626227668133590597632})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/77371252455336267181195264})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/154742504910672534362390528})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/309485009821345068724781056})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/618970019642690137449562112})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1237940039285380274899124224})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2475880078570760549798248448})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/4951760157141521099596496896})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/9903520314283042199192993792})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/19807040628566084398385987584})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/39614081257132168796771975168})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/79228162514264337593543950336})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/158456325028528675187087900672})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/316912650057057350374175801344})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/633825300114114700748351602688})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1267650600228229401496703205376})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2535301200456458802993406410752})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/5070602400912917605986812821504})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/10141204801825835211973625643008})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/20282409603651670423947251286016})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/40564819207303340847894502572032})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/81129638414606681695789005144064})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/162259276829213363391578010288128})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/324518553658426726783156020576256})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/649037107316853453566312041152512})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1298074214633706907132624082305024})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2596148429267413814265248164610048})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/5192296858534827628530496329220096})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/10384593717069655257060992658440192})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/20769187434139310514121985316880384})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/41538374868278621028243970633760768})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/83076749736557242056487941267521536})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/166153499473114484112975882535043072})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/332306998946228968225951765070086144})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/664613997892457936451903530140172288})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1329227995784915872903807060280344576})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2658455991569831745807614120560689152})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/5316911983139663491615228241121378304})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/10633823966279326983230456482242756608})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/21267647932558653966460912964485513216})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/42535295865117307932921825928971026432})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/85070591730234615865843651857942052864})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/170141183460469231731687303715884105728})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/340282366920938463463374607431768211456})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/680564733841876926926749214863536422912})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1361129467683753853853498429727072845824})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2722258935367507707706996859454145691648})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/5444517870735015415413993718908291383296})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/10889035741470030830827987437816582766592})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/21778071482940061661655974875633165533184})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/43556142965880123323311949751266331066368})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/87112285931760246646623899502532662132736})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/174224571863520493293247799005065324265472})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/348449143727040986586495598010130648530944})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/696898287454081973172991196020261297061888})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1393796574908163946345982392040522594123776})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2787593149816327892691964784081045188247552})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/5575186299632655785383929568162090376495104})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/11150372599265311570767859136324180752990208})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/22300745198530623141535718272648361505980416})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/44601490397061246283071436545296723011960832})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/89202980794122492566142873090593446023921664})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/178405961588244985132285746181186892047843328})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/356811923176489970264571492362373784095686656})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/713623846352979940529142984724747568191373312})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1427247692705959881058285969449495136382746624})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2854495385411919762116571938898990272765493248})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/5708990770823839524233143877797980545530986496})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/11417981541647679048466287755595961091061972992})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/22835963083295358096932575511191922182123945984})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/45671926166590716193865151022383844364247891968})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/91343852333181432387730302044767688728495783936})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/182687704666362864775460604089535377456991567872})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/365375409332725729550921208179070754913983135744})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/730750818665451459101842416358141509827966271488})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1461501637330902918203684832716283019655932542976})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2923003274661805836407369665432566039311865085952})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/5846006549323611672814739330865132078623730171904})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/11692013098647223345629478661730264157247460343808})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/23384026197294446691258957323460528314494920687616})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/46768052394588893382517914646921056628989841375232})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/93536104789177786765035829293842113257979682750464})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/187072209578355573530071658587684226515959365500928})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/374144419156711147060143317175368453031918731001856})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/748288838313422294120286634350736906063837462003712})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1496577676626844588240573268701473812127674924007424})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/2993155353253689176481146537402947624255349848014848})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/5986310706507378352962293074805895248510699696029696})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/11972621413014756705924586149611790497021399392059392})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/23945242826029513411849172299223580994042798784118784})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/47890485652059026823698344598447161988085597568237568})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/95780971304118053647396689196894323976171195136475136})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/191561942608236107294793378393788647952342390272950272})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/383123885216472214589586756787577295904684780545900544})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/766247770432944429179173513575154591809369561091801088})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1532495540865888858358347027150309183618739122183602176})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/3064991081731777716716694054300618367237478244367204352})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/6129982163463555433433388108601236734474956488734408704})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/12259964326927110866866776217202473468949912977468817408})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/24519928653854221733733552434404946937899825954937634816})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/49039857307708443467467104868809893875799651909875269632})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/98079714615416886934934209737619787751599303819750539264})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/196159429230833773869868419475239575503198607639501078528})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/392318858461667547739736838950479151006397215279002157056})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/784637716923335095479473677900958302012794430558004314112})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1569275433846670190958947355801916604025588861116008628224})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/3138550867693340381917894711603833208051177722232017256448})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/6277101735386680763835789423207666416102355444464034512896})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/12554203470773361527671578846415332832204710888928069025792})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/25108406941546723055343157692830665664409421777856138051584})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/50216813883093446110686315385661331328818843555712276103168})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/100433627766186892221372630771322662657637687111424552206336})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/200867255532373784442745261542645325315275374222849104412672})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/401734511064747568885490523085290650630550748445698208825344})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/803469022129495137770981046170581301261101496891396417650688})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1606938044258990275541962092341162602522202993782792835301376})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/3213876088517980551083924184682325205044405987565585670602752})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/6427752177035961102167848369364650410088811975131171341205504})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/12855504354071922204335696738729300820177623950262342682411008})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/25711008708143844408671393477458601640355247900524685364822016})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/51422017416287688817342786954917203280710495801049370729644032})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/102844034832575377634685573909834406561420991602098741459288064})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/205688069665150755269371147819668813122841983204197482918576128})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/411376139330301510538742295639337626245683966408394965837152256})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/822752278660603021077484591278675252491367932816789931674304512})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/1645504557321206042154969182557350504982735865633579863348609024})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/3291009114642412084309938365114701009965471731267159726697218048})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/6582018229284824168619876730229402019930943462534319453394436096})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/13164036458569648337239753460458804039861886925068638906788872192})^{1/2} = o(e^{0.5-\epsilon}(F^*)^{1/263280729171392966$$

**Proof of Theorem 1.**

$$Q(F^*)R(F^*).$$

$$\begin{matrix} A & Q(F^*) & R(F^*), \\ Q(F^*)R(F^*) & , & : \end{matrix}$$

$$\begin{aligned} Q(F^*)R(F^*) &= \text{diag}[\text{diag}[\mathcal{A}], a_1^{*-1}P_1, \dots, a_K^{*-1}P_K] \\ &\quad \times \begin{pmatrix} R_{b,b} & R_{b,G_1} & \cdots & R_{b,G_K} \\ R_{G_1,b} & R_{G_1,G_1} & \cdots & R_{G_1,G_K} \\ \vdots & \vdots & \ddots & \vdots \\ R_{G_K,b} & R_{G_K,G_1} & \cdots & R_{G_K,G_K} \end{pmatrix} \\ &= \begin{pmatrix} \text{diag}[\mathcal{A}]R_{b,b} & \text{diag}[\mathcal{A}]R_{b,G_1} & \cdots & \text{diag}[\mathcal{A}]R_{b,G_K} \\ a_1^{*-1}P_1R_{G_1,b} & a_1^{*-1}P_1R_{G_1,G_1} & \cdots & a_1^{*-1}P_1R_{G_1,G_K} \\ \vdots & \vdots & \ddots & \vdots \\ a_K^{*-1}P_KR_{G_K,b} & a_K^{*-1}P_KR_{G_K,G_1} & \cdots & a_K^{*-1}P_KR_{G_K,G_K} \end{pmatrix}, \end{aligned}$$

$$\begin{matrix} b(x) = [b_1(x), \dots, b_K(x)] & \mathcal{A} = [a_1^*, \dots, a_K^*] \\ R(F^*) & V(x) \end{matrix}.$$

$$V(x) = [b(x), a_1^*b_1(x)G_1(x), \dots, a_K^*b_K(x)G_K(x)].$$

$$\begin{matrix} ( ) \text{ The computation of } \text{diag}[\mathcal{A}]R_{b,b} : F & b_i(x) \\ h_i(x) = a_i^*b_i(x), & \end{matrix}$$

$$\begin{aligned} \int_{R^n} b_i(x)b_j(x)P(x|F^*) \, m &= \frac{1}{a_i^*a_j^*} e_{ij}(F^*) \quad i \neq j, \\ \int_{R^n} b_i^2(x)P(x|F^*) \, m &= \frac{1}{a_i^*} - \frac{1}{(a_i^*)^2} e_{ii}(F^*) \end{aligned}$$

$$\text{diag}[\mathcal{A}]R_{b,b} = I_K + \begin{pmatrix} -a_1^{*-1}e_{11}(F^*) & a_2^{*-1}e_{12}(F^*) & \cdots & a_K^{*-1}e_{1K}(F^*) \\ a_1^{*-1}e_{21}(F^*) & -a_2^{*-1}e_{22}(F^*) & \cdots & a_K^{*-1}e_{2K}(F^*) \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{*-1}e_{K1}(F^*) & a_2^{*-1}e_{K2}(F^*) & \cdots & -a_K^{*-1}e_{KK}(F^*) \end{pmatrix}.$$

B

$$\frac{1}{a_j^*} e_{ij}(F^*) \leq \frac{1}{a} e_{ij}(F^*) = o(0.5^{-e}(F^*)),$$

$$diag[\mathcal{A}]R_{\mathbf{b},\mathbf{b}}=I_K+o$$

$$\begin{aligned}
& \text{C} \quad , \quad |g_{ij}(x)| \leq 1 \quad . \text{I} \\
& |E(h_j(X)(h_i(X) - d_{ij})(t_{i,k}(X) - f_{i,k}^*))| \\
& \leq E(|h_j(X)(h_i(X) - d_{ij})|(t_{i,k}(X) - f_{i,k}^*)) \\
& \leq E^{1/2}(g_{ij}^2(X))E^{1/2}((t_{i,k}(X) - f_{i,k}^*)^2) \\
& \leq E^{1/2}(|g_{ij}(X)|)E^{1/2}((t_{i,k}(X) - f_{i,k}^*)^2) \\
& \leq \sqrt{e_{ij}(\mathbf{F}^*)}E^{1/2}((t_{i,k}(X) - f_{i,k}^*)^2). \\
& \text{A} \quad \text{L} \quad 2, E(\|t_i(X) - f_i^*\|^2|\mathbf{F}^*) \quad \mathbf{u}M_i^q(\mathbf{F}^*). \\
& \quad E^{1/2}((t_{i,k}(X) - f_{i,k}^*)^2) \quad \sqrt{\mathbf{u}M_i^q(\mathbf{F}^*)}. \\
& \quad , \\
& E(\text{diag}[\mathcal{A}] \mathbf{a}_i^* \mathbf{b}_i(X) \mathbf{b}(X)(t_i(X) - f_i^*)) = O(M_i^{q/2}(\mathbf{F}^*)e^{0.5}(\mathbf{F}^*)). \\
& \text{A} \quad \text{L} \quad 1 \quad 3, M_i^{q/2}(\mathbf{F}^*)^{0.5}(\mathbf{F}^*) \quad e(\mathbf{F}^*) \\
& \text{Z}(\mathbf{F}^*) \quad . \text{I} \\
& \|E(\text{diag}[\mathcal{A}] \mathbf{a}_i^* \mathbf{b}_i(X) \mathbf{b}(X)(t_i(X) - f_i^*))\| = O(M_i^{q/2}(\mathbf{F}^*)^{0.5}(\mathbf{F}^*)). \\
& \text{M} \quad , \\
& \|\text{diag}[\mathcal{A}] \mathbf{R}_{\mathbf{b}, \mathbf{G}_i}\| \leq \|E(\text{diag}[\mathcal{A}] \mathbf{a}_i^* \mathbf{b}_i(X) \mathbf{b}(X)(t_i(X) - f_i^*))\| \|\mathbf{P}_i^{-1}\| \\
& \|\mathbf{P}_i^{-1}\| = \|I(\mathbf{f}_i^*)\| \leq O(\|m_i^*\|^{\mathbf{t}_1}(1^i)^{-\mathbf{t}_2}) \\
& \text{C} \quad (4). \quad , \\
& \|\text{diag}[\mathcal{A}] \mathbf{R}_{\mathbf{b}, \mathbf{G}_i}\| \leq \mathbf{u} \|m_i^* - m_j^*\|^{q_1} (1^i)^{-q_2} {}^{0.5}(\mathbf{F}^*), \\
& q_1 = (q/2) + \mathbf{t}_1, q_2 = \mathbf{t}_2, \quad \mathbf{u} \\
& \quad , \quad \text{L} \quad 1 \quad 3 \\
& \|\text{diag}[\mathcal{A}] \mathbf{R}_{\mathbf{b}, \mathbf{G}_i}\| \leq O(Z^{-q_1 \vee q_2}(\mathbf{F}^*))e^{0.5}(\mathbf{F}^*) = o(e^{0.5-\mathbf{e}}(\mathbf{F}^*)). \\
& \text{B} \quad , \\
& \text{diag}[\mathcal{A}] \mathbf{R}_{\mathbf{b}, \mathbf{G}_i} = o({}^{0.5-\mathbf{e}}(\mathbf{F}^*)). \\
& ( ) \text{ The computation of } \mathbf{a}_i^{*-1} \mathbf{P}_i \mathbf{R}_{\mathbf{G}_i, \mathbf{b}} \quad (i = 1, \dots, K): \text{A} \quad (1) \\
& \text{L} \quad 2, \mathbf{a}_i^{*-1} \|\mathbf{P}_i\| \quad (1/\mathbf{a}) \mathbf{c} \|m_i^* - m_j^*\|^p, \quad j \neq i, \mathbf{c} \quad p \\
& \quad . \text{B} \quad \mathbf{R}_{\mathbf{G}_i, \mathbf{b}} = \mathbf{R}_{\mathbf{b}, \mathbf{G}_i}, \quad ( ) \quad : \\
& \mathbf{a}_i^{*-1} \mathbf{P}_i \mathbf{R}_{\mathbf{G}_i, \mathbf{b}} = o({}^{0.5-\mathbf{e}}(\mathbf{F}^*)).
\end{aligned}$$

( ) The computation of  $\mathbf{a}_i^{*-1} \mathbf{P}_i R_{\mathbf{G}_i, \mathbf{G}_i}$  ( $i = 1, \dots, K$ ): B  $V(x)$ ,

$$\begin{aligned} \mathbf{a}_i^{*-1} \mathbf{P}_i R_{\mathbf{G}_i, \mathbf{G}_i} &= \mathbf{a}_i^{*-1} \mathbf{P}_i E(h_i^2(X) \mathbf{G}_i(X) \mathbf{G}_i(X)) \\ &= \mathbf{a}_i^{*-1} E(h_i^2(X)(t_i(X) - \mathbf{f}_i^*)(t_i(X) - \mathbf{f}_i^*)) \mathbf{P}_i^{-1} \\ &= I_{d_i} + \mathbf{a}_i^{*-1} E(h_i(X)(h_i(X) - 1)(t_i(X) - \mathbf{f}_i^*)(t_i(X) - \mathbf{f}_i^*)) \mathbf{P}_i^{-1}, \\ &\quad \vdots \\ \mathbf{P}_i E(h_i(X) \mathbf{G}_i(X) \mathbf{G}_i(X)) &= \mathbf{a}_i^* I_{d_i}. \end{aligned}$$

$$\begin{aligned} \mathbf{F} \quad \mathbf{a}_i^{*-1} \quad , \quad E(\|t_i(X) - \mathbf{f}_i^*\|^2 | \mathbf{F}^*) \quad , \quad \mathbf{u} M_i^q(\mathbf{F}^*) \\ \mathbf{a}_i^{*-1} E(h_i(X)(h_i(X) - 1)(t_i(X) - \mathbf{f}_i^*)(t_i(X) - \mathbf{f}_i^*)) \mathbf{P}_i^{-1} = o(\epsilon^{0.5-\epsilon}(\mathbf{F}^*)) \\ , \\ \mathbf{a}_i^{*-1} \mathbf{P}_i R_{\mathbf{G}_i, \mathbf{G}_i} = I_{d_i} + o(\epsilon^{0.5-\epsilon}(\mathbf{F}^*)). \end{aligned}$$

( ) The computation of  $\mathbf{a}_i^{*-1} \mathbf{P}_i R_{\mathbf{G}_i, \mathbf{G}_j}$  ( $j \neq i$ ): B  $V(x)$ ,

$$\begin{aligned} \mathbf{a}_i^{*-1} \mathbf{P}_i R_{\mathbf{G}_i, \mathbf{G}_j} &= \mathbf{a}_i^{*-1} E(\mathbf{a}_i^* \mathbf{b}_i(X) \mathbf{a}_j^* \mathbf{b}_j(X)(t_i(X) - \mathbf{f}_i^*)(t_j(X) - \mathbf{f}_j^*)) \mathbf{P}_j^{-1} \\ &= \mathbf{a}_i^{*-1} E(h_i(X) h_j(X)(t_i(X) - \mathbf{f}_i^*)(t_j(X) - \mathbf{f}_j^*)) \mathbf{P}_j^{-1}. \end{aligned}$$

$$( ), \quad \mathbf{a}_i^{*-1} \mathbf{P}_i R_{\mathbf{G}_i, \mathbf{G}_j} = o(\epsilon^{0.5-\epsilon}(\mathbf{F}^*)).$$

$$( ) ( ), \quad : \quad Q(\mathbf{F}^*) R(\mathbf{F}^*) = I + o(\epsilon^{0.5-\epsilon}).$$

$$, \quad \mathbf{E} \quad . \quad (12),$$

$$r \leq \|I - Q(\mathbf{F}^*) R(\mathbf{F}^*)\| = o(\epsilon^{0.5-\epsilon}(\mathbf{F}^*)). \quad \square$$

#### 4. A typical class: Gaussian mixtures

$$\begin{aligned} \text{EM} \\ \mathbf{G} \quad . \quad \mathbf{A} \quad 1, \quad \mathbf{G} \quad y_i = \frac{P_i(x|m_i, \mathbf{S}_i)}{(\mathbf{S}_i^{-1} m_i, \mathbf{S}_i^{-1})} \quad \mathbf{E} \quad . \quad (2) \\ t_i(x) = (x, -\frac{1}{2} x x) \quad , \quad \mathbf{f}_i, \quad y_i, \\ (m_i, -\frac{1}{2}(\mathbf{S}_i + m_i m_i)) \quad (m_i, \mathbf{S}_i) \end{aligned}$$

$N$  ,  $G$

**Lemma 4.** Suppose that  $P_i(x|f_i^*) = P_i(x|m_i^*, S_i^*)$  is a Gaussian distribution with the mean  $m_i^*$  and the covariance matrix  $S_i^*$ , and that the condition number of  $S_i^*$ , i.e.,  $k(S_i^*)$ , is upper bounded by  $B'$ . We have that  $P_i(x|\hat{f}_i^*)$  is bell-sheltered, i.e.,

$$P_i(x|f_i^*) = P_i(x|m_i^*, S_i^*) \leq b \frac{1}{(l^i)^{n/2}} e^{-(1/2l^i) \|x - m_i^*\|^2}, \quad (29)$$

where  $b$  is a positive number.

**Proof.**  $y = U_i(x - m_i^*)$

$$P(y|l^i) = \frac{1}{(2pl^i)^{n/2}} e^{-(1/2l^i) \|y\|^2},$$

$$P_i(x|m_i^*, S_i^*) \leq B'^{n/2} P(y|l^i),$$

$$k(S_i^*) \leq B'. \quad M, \quad \|y\| = \|x - m_i^*\|,$$

$$P_i(x|m_i^*, S_i^*) \leq b \frac{1}{(l^i)^{n/2}} e^{-(1/2l^i) \|x - m_i^*\|^2},$$

$$b = (B'/2p)^{n/2}. \quad \square$$

$B, L, 4, (1) (3), G, F^*$

$M, K$

$P_i(x|f_i^*) = P_i(x|m_i^*, S_i^*), \quad t_i(x)$

:

$$t_i(x) = \begin{cases} x & m_i^*, \\ -\frac{1}{2}xx & -\frac{1}{2}(S_i^* + m_i^*(m_i^*)). \end{cases}$$

$$, \quad G, \quad \begin{matrix} t_i(x) \\ F^* \end{matrix} \quad \begin{matrix} x_1, \dots, x_n. \\ (1) (3) \end{matrix} \quad (4).$$

$F, G, (4), (2),$

$$f_i^* = [(m_i^*), \text{vec}[S_i^*]]^T, \quad \dot{S}_i^* = -\frac{1}{2}(S_i^* + m_i^*(m_i^*)), \quad \hat{f}_i^* = [(m_i^*), \text{vec}[S_i^*]]^T.$$

**Lemma 5.** Suppose that  $P_i(x|f_i^*) = P_i(x|m_i^*, S_i^*)$  is a Gaussian density and  $k(S_i^*)$  is upper bounded. As  $l^i$  tends to zero, we have

$$\|I(f_i^*)\| = O(l^i)^{-t}, \quad (30)$$

where  $t$  is a positive number.

**Proof.** B

$$\frac{\partial P_i(x|m_i^*, \mathbf{S}_i^*)}{\partial m_i^*} = (x - m_i^*) \mathbf{S}_i^* P_i(x|m_i^*, \mathbf{S}_i^*), \quad (31)$$

$$\frac{\partial P_i(x|m_i^*, \mathbf{S}_i^*)}{\partial \mathbf{S}_i^*} = -\frac{1}{2}(\mathbf{S}_i^{*-1} - \mathbf{S}_i^{*-1}(x - m_i^*)(x - m_i^*)^T \mathbf{S}_i^{*-1}) P_i(x|m_i^*, \mathbf{S}_i^*). \quad (32)$$

A

F

$$\begin{aligned} I(\mathbf{f}_i^*) &= E_{\mathbf{f}_i^*} \left( \left( \frac{\partial P_i(X|\mathbf{f}_i^*)}{\partial \mathbf{f}_i^*} \right) \left( \frac{\partial P_i(X|\mathbf{f}_i^*)}{\partial \mathbf{f}_i^*} \right)^T \right) \\ &= E_{\mathbf{f}_i^*} \left( \left( \frac{\partial \hat{\mathbf{f}}_i^*}{\partial \mathbf{f}_i^*} \right) \left( \frac{\partial P_i(X|\mathbf{f}_i^*)}{\partial \hat{\mathbf{f}}_i^*} \right) \left( \frac{\partial P_i(X|\mathbf{f}_i^*)}{\partial \hat{\mathbf{f}}_i^*} \right)^T \left( \frac{\partial \hat{\mathbf{f}}_i^*}{\partial \mathbf{f}_i^*} \right)^T \right) \\ &= \frac{\partial \hat{\mathbf{f}}_i^*}{\partial \mathbf{f}_i^*} I(\hat{\mathbf{f}}_i^*) \left( \frac{\partial \hat{\mathbf{f}}_i^*}{\partial \mathbf{f}_i^*} \right)^T, \end{aligned}$$

$$I(\hat{\mathbf{f}}_i^*) = E_{\hat{\mathbf{f}}_i^*} \left( \left( \frac{\partial P_i(X|\mathbf{f}_i^*)}{\partial \hat{\mathbf{f}}_i^*} \right) \left( \frac{\partial P_i(X|\mathbf{f}_i^*)}{\partial \hat{\mathbf{f}}_i^*} \right)^T \right).$$

I E . (31)

(32)

$$\begin{aligned} &P_i^3(x|m_i^*, \mathbf{S}_i^*) \\ &P_i(x|m_i^*, \frac{1}{3}\mathbf{S}_i^*) \end{aligned}$$

$$I(\hat{\mathbf{f}}_i^*)$$

G

$$|\mathbf{S}_i^*|$$

$$I(\hat{\mathbf{f}}_i^*) = E_{(m_i^*, (1/3)\mathbf{S}_i^*)}(G(X, \mathbf{f}_i^*)),$$

$$\begin{aligned} &G(x, \mathbf{f}_i^*) \\ y = x - m_i^*, \end{aligned} \quad x - m_i^* \quad \mathbf{S}_i^*.$$

$$I(\hat{\mathbf{f}}_i^*) = E_{(0, (1/3)\mathbf{S}_i^*)}(G(Y, \mathbf{S}_i^*)),$$

$$\begin{aligned} &G(y, \mathbf{S}_i^*) \\ y_1, \dots, y_n. \end{aligned} \quad \mathbf{S}_i^{*-1} \quad g_{pq}(y, \mathbf{S}_i^*)$$

$$\mathbf{S}_i^{*-1} = |\mathbf{S}_i^*|^{-1} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d_i} \\ a_{21} & a_{22} & \cdots & a_{2d_i} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d_i1} & a_{d_i2} & \cdots & a_{d_id_i} \end{pmatrix},$$

$$a_{kl}$$

$$g_{pq}(y, \mathbf{S}_i^*)$$

$$|\mathbf{S}_i^*|.$$

$$\mathbf{s}_{kl}^{*i}$$

$$\mathbf{s}_{kl}^{*i}$$

$$\mathbf{S}_i^*,$$

$$\mathbf{s}_{kl}^{*j}$$



$$\begin{aligned}\|I(\hat{\mathbf{f}}_i^*)\| &= \|(\mathbf{1}^i)^{-\mathbf{t}}(\mathbf{1}^i)^{\mathbf{t}}I(\mathbf{f}_i^*)\| \\ &= (\mathbf{1}^i)^{-\mathbf{t}}\|(\mathbf{1}^i)^{\mathbf{t}}I(\mathbf{f}_i^*)\| \\ &\leq o(\mathbf{1}^i)^{-\mathbf{t}},\end{aligned}$$

$$\begin{aligned} \|I(\hat{\mathbf{f}}_i^*)\| &\leq \left\| \frac{\partial(\hat{\mathbf{f}}_i^*)}{\partial \mathbf{f}_i^*} \right\| \|I(\hat{\mathbf{f}}_i^*)\| \left\| \left( \frac{\partial(\hat{\mathbf{f}}_i^*)}{\partial \mathbf{f}_i^*} \right) \right\| = 4 \|I(\hat{\mathbf{f}}_i^*)\|, \\ \|\partial(\hat{\mathbf{f}}_i^*) / \partial \mathbf{f}_i^*\| &= \|\partial(\hat{\mathbf{f}}_i^*) / \partial \mathbf{f}_i^*\| = 2, \end{aligned}$$

EM (1) (3)  
G

$$\|G'(\mathbf{F}^*)\| = o(e^{0.5-\epsilon(\mathbf{F}^*)}), \quad (33)$$
$$\frac{I}{(1+I)^3} = \frac{1}{1+G} \quad (1)$$

I

, EM

-N M

H K A (P C G HK4225/04E) G G C N  
F C (P . 60071004).

## Appendix

### Proof of Lemma 1.

$$Z(F^*) = \sum_{i \neq j} Z_i(m_j^*) = Z_i(m_j^*). \quad (2) \quad (3),$$

$$a_1(1^{i'})^n \leq (1^i)^n \leq a_2(1^{i'})^n, \quad (34)$$

$$b_1(1^i)^n \leq (1^j)^n \leq b_2(1^i)^n, \quad (35)$$

$$c_1 \|m_{i'}^* - m_j^*\| \leq \|m_i^* - m_j^*\| \leq c_2 \|m_{i'}^* - m_j^*\|. \quad (36)$$

$$C \quad E \quad (34) \quad (35) \quad E \quad (36),$$

$$a'_1, a'_2, b'_1, b'_2$$

$$a'_1 Z(F^*) \leq Z_i(m_j^*) \leq a'_2 Z(F^*),$$

$$b'_1 Z_i(m_j^*) \leq Z_j(m_i^*) \leq b'_2 Z_i(m_j^*).$$

$$\begin{aligned} & \quad , Z(F^*), Z_i(m_j^*) \quad Z_j(m_i^*) \\ & \quad ( ), \|m_i^* - m_j^*\| \\ & \quad \cdot I \quad \|m_i^* - m_j^*\| \\ & \|m_i^*\| \leq \|m_i^* - m_j^*\| + \|m_j^*\|, E \quad (23) \\ & \quad \cdot O \quad , \|m_i^*\| \quad \|m_j^*\| \\ & \quad \|m_i^*\| \\ & E \quad (23). \quad , ( ) \end{aligned}$$

$$F \quad , \quad ( )$$

$$I \quad p = q > 0, \quad ( ),$$

$$\begin{aligned} \|m_i^* - m_j^*\|^p (1^i)^{-nq} &= \|m_i^* - m_j^*\|^p (1^i)^{-np} \\ &= (Z_i(m_j^*))^{-p} = O(Z^{-p}(F^*)) = O(Z^{-p \vee q}(F^*)). \end{aligned}$$

$$I \quad p > q, \quad 1^i \quad ( ),$$

$$\|m_i^* - m_j^*\|^p (1^i)^{-nq} \leq O(Z^{-p}(F^*)) = O(Z^{-p \vee q}(F^*)).$$

$$I \quad p < q, \quad \|m_i^* - m_j^*\| \geq T,$$

$$\|m_i^* - m_j^*\|^p (1^i)^{-nq} \leq O(Z^{-q}(F^*)) = O(Z^{-p \vee q}(F^*)).$$

$$\|m_i^* - m_j^*\|^p (1^i)^{-nq} \leq O(Z^{-p \vee q}(F^*)). \quad \square$$

### Proof of Lemma 2.

$$( ). A$$

$$\|P_i\| = \|E_{f_i^*}((t_i(X) - f_i^*)(t_i(X) - f_i^*))\| \leq E_{f_i^*}(\|t_i(X) - f_i^*\|^2). \quad (37)$$

$$t_i(x) \quad x_1, x_2, \dots, x_n, \dots, x,$$

$$t_i(x) = P_0 + P_1 x + P_2 x^2 + \dots + P_k x^k,$$

$$k \geq 0, \quad P_i \quad d_i \times n^i \quad , \quad x^i$$

$$x_1, x_2, \dots, x_n. \quad \mathbf{I} \quad \quad \quad \|x^i\| \leq \sqrt{n} \|x\|^i \quad i = 0, 1, \dots, k.$$

$$\mathbf{B} \quad , \quad \quad \quad x_{j_1} x_{j_2} \cdots x_{j_p}$$

$$t_i(x) = t_i(x - m_i^* + m_i^*)$$

$$= P'_0 + P'_1(x - m_i^*) + P'_2(x - m_i^*)^2 + \cdots + P'_k(x - m_i^*)^k, \quad (38)$$

$$P'_i \quad d_i \times n^i \quad , \quad m_{i1}^*, \dots, m_{in}^*.$$

$$\mathbf{f}_i^* = E_{\mathbf{f}_i^*}(t_i(X)) = P'_0 + E_{\mathbf{f}_i^*}(P'_1(X - m_i^*)) + \cdots + E_{\mathbf{f}_i^*}(P'_k(X - m_i^*)^k) \quad (39)$$

$$E_{\mathbf{f}_i^*}(P'_1(X - m_i^*)) = P'_1 E_{\mathbf{f}_i^*}(X - m_i^*) = 0,$$

$$t_i(X) - \mathbf{f}_i^* = \sum_{j=1}^k [P'_j(X - m_i^*)^j - E_{\mathbf{f}_i^*}(P'_j(X - m_i^*)^j)].$$

$\mathbf{N}$  ,

$$E_{\mathbf{f}_i^*}(\|t_i(X) - \mathbf{f}_i^*\|^2) = E_{\mathbf{f}_i^*}(\|(t_i(X) - \mathbf{f}_i^*) \quad (t_i(X) - \mathbf{f}_i^*)\|)$$

$$= E_{\mathbf{f}_i^*} \left( \left\| \sum_{j_1=1, j_2=1}^k [P'_{j_1}(X - m_i^*)^{j_1} - E_{\mathbf{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})] \right. \right.$$

$$\times [P'_{j_2}(X - m_i^*)^{j_2} - E_{\mathbf{f}_i^*}(P'_{j_2}(X - m_i^*)^{j_2})] \left. \right\|$$

$$\leq \sum_{j_1=1, j_2=1}^k E_{\mathbf{f}_i^*}(\| [P'_{j_1}(X - m_i^*)^{j_1} - E_{\mathbf{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})] \quad \|$$

$$\times \| [P'_{j_2}(X - m_i^*)^{j_2} - E_{\mathbf{f}_i^*}(P'_{j_2}(X - m_i^*)^{j_2})] \|)$$

$$\leq \sum_{j_1=1, j_2=1}^k E_{\mathbf{f}_i^*}^{1/2}(\|P'_{j_1}(X - m_i^*)^{j_1} - E_{\mathbf{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})\|^2)$$

$$\times E_{\mathbf{f}_i^*}^{1/2}(\|P'_{j_2}(X - m_i^*)^{j_2} - E_{\mathbf{f}_i^*}(P'_{j_2}(X - m_i^*)^{j_2})\|^2 | \mathbf{f}_i^* \|)^2. \quad (40)$$

$\mathbf{P}$  ,

$$E_{\mathbf{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1} - E_{\mathbf{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})\|^2)$$

$$= E_{\mathbf{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1}\|^2) - \|E_{\mathbf{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1})\|^2$$

$$\leq E_{\mathbf{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1}\|^2) \leq \sqrt{n} E_{\mathbf{f}_i^*}(\|P'_{j_1}\|^2 \|X - m_i^*\|^{2j_1})$$

$$= \sqrt{n} \|P'_{j_1}\|^2 E_{\mathbf{f}_i^*}(\|X - m_i^*\|^{2j_1}). \quad (41)$$



$$\begin{aligned}
\text{A} \quad & \quad \quad \quad j \neq i, \quad \quad \quad \mathbf{f}'_j = E_{\mathbf{f}_j^*}(t_i(X)) \\
& E_{\mathbf{f}_j^*}(\|t_i(X) - \mathbf{f}_i^*\|^2) \leq E_{\mathbf{f}_j^*}((\|t_i(X) - \mathbf{f}'_j\| + \|\mathbf{f}'_j - \mathbf{f}_i^*\|)^2) \\
& \quad \quad \quad = E_{\mathbf{f}_j^*}(\|t_i(X) - \mathbf{f}'_j\|^2 + 2\|t_i(X) - \mathbf{f}'_j\|\|\mathbf{f}'_j - \mathbf{f}_i^*\| + \|\mathbf{f}'_j - \mathbf{f}_i^*\|^2) \\
& \quad \quad \quad \leq E_{\mathbf{f}_j^*}(2\|t_i(X) - \mathbf{f}'_j\|^2 + 2\|\mathbf{f}'_j - \mathbf{f}_i^*\|^2) \\
& \quad \quad \quad = 2E_{\mathbf{f}_j^*}(\|t_i(X) - \mathbf{f}'_j\|^2) + 2\|\mathbf{f}_i^* - \mathbf{f}'_j\|^2. \tag{48}
\end{aligned}$$

$$\begin{aligned} & E_{\mathbf{f}_j^*}(\|t_i(X) - \mathbf{f}'_j\|^2) \leq c_1 \|m_i^* - m_j^*\|^{p_1}, \\ & \quad c_1 = p_1 \cdot M, \\ & \quad \|\mathbf{f}_i^* - \mathbf{f}'_j\| \leq \|\mathbf{f}_i^*\| + \|\mathbf{f}'_j\|. \end{aligned} \quad (49)$$

$$E_{f_i^*}(\|t_i(X) - f_i^*\|^2) \leq c_j \|m_i^* - m_j^*\|^{p_j}, \quad j \neq i, \quad (50)$$

$$B = \begin{pmatrix} C_j & p_j \\ E & \end{pmatrix}. \quad (47) \quad (50),$$

$$E(\|t_i(X) - \mathbf{f}_i^*\|^2) = \sum_{j=1}^K \mathbf{a}_j^* E_{\mathbf{f}_j^*}(\|t_i(X) - \mathbf{f}_i^*\|^2) \leq \mathbf{u} M_i^q(\mathbf{F}^*),$$

$$M_i(F^*) = \sum_{j \neq i} \|m_i^* - m_j^*\|, \mathbf{u} \quad q \quad . \quad \square$$

### Proof of Lemma 3.

$$f(\mathbf{Z}) = o(Z^p),$$

$$\mathbf{Z} \rightarrow 0,$$

$$p$$

$$K$$

$$\mathbf{F}^*$$

$$\mathbf{Z}(\mathbf{F}^*) = \mathbf{Z}.$$

$$i \neq j,$$

$$\mathbf{Z},$$

$$m_{ij}^*$$

$$m_i^*$$

$$m_j^*$$

$$\mathbf{a}_i^* P_i(m_{ij}^* | \mathbf{f}_i^*) = \mathbf{a}_j^* P_j(m_{ij}^* | \mathbf{f}_j^*).$$

$$E_i = \{x : \mathbf{a}_i^* P_i(x|\mathbf{f}_i^*) \geq \mathbf{a}_i^* P_j(x|\mathbf{f}_j^*)\},$$

$$E_j = \{x : \mathbf{a}_j^* P_j(x|\mathbf{f}_j^*) > \mathbf{a}_i^* P_i(x|\mathbf{f}_i^*)\}.$$

$$\mathbf{A} = \mathbf{Z}(\mathbf{F}^*) \begin{pmatrix} \mathbf{M}_i^* & \mathbf{M}_j^* \\ \mathbf{k}(\mathbf{S}_i^*) & \mathbf{k}(\mathbf{S}_j^*) \end{pmatrix} \begin{pmatrix} (\mathbf{I}^i)^n / (\|\mathbf{m}_i^* - \mathbf{m}_j^*\|) & (\mathbf{I}^j)^n / (\|\mathbf{m}_i^* - \mathbf{m}_j^*\|) \\ \mathcal{N}_{r_i}(\mathbf{m}_i^*) & \mathcal{N}_{r_j}(\mathbf{m}_j^*) \end{pmatrix} \begin{pmatrix} \mathbf{m}_i^* & \mathbf{m}_j^* \\ \mathbf{E}_i & \mathbf{E}_j \end{pmatrix} \mathbf{F} \begin{pmatrix} \mathbf{k}(\mathbf{S}_i^*) & \mathbf{k}(\mathbf{S}_j^*) \end{pmatrix} \begin{pmatrix} \mathbf{E}_i & \mathbf{E}_j \end{pmatrix}.$$

$$, r_i \quad r_j \quad \|m_i^* - m_j^*\| \quad \|m_i^* - m_j^*\| b_1$$

$b_2$

$$r_i \geq b_i \|m_i^* - m_j^*\| \quad r_j \geq b_j \|m_i^* - m_j^*\|.$$

$$\mathcal{D}_i = \mathcal{N}_{r_i}^c(m_i^*) = \{x: \|x - m_i^*\| \geq r_i\},$$

$$\mathcal{D}_j = \mathcal{N}_{r_j}^c(m_j^*) = \{x: \|x - m_j^*\| \geq r_j\}$$

$$E_i \subset D_j, \quad E_j \subset D_i.$$

$$\mathbf{M} \quad , \quad e_{ij}(\mathbf{F}^*) \quad h_k(x)$$

$$\begin{aligned} e_{ij}(\mathbf{F}^*) &= \int h_i(x) h_j(x) P(x|\mathbf{F}^*) \, \mathbf{m} \\ &= \int_{E_i} h_i(x) h_j(x) P(x|\mathbf{F}^*) \, \mathbf{m} + \int_{E_j} h_i(x) h_j(x) P(x|\mathbf{F}^*) \, \mathbf{m} \\ &\leq \int_{\mathcal{D}_j} h_i(x) h_j(x) P(x|\mathbf{F}^*) \, \mathbf{m} + \int_{\mathcal{D}_i} h_i(x) h_j(x) P(x|\mathbf{F}^*) \, \mathbf{m} \\ &\leq \int_{\mathcal{D}_j} h_j(x) P(x|\mathbf{F}^*) \, \mathbf{m} + \int_{\mathcal{D}_i} h_i(x) P(x|\mathbf{F}^*) \, \mathbf{m} \\ &= \mathbf{a}_j^* \int_{\mathcal{D}_j} P_j(x|\mathbf{f}_j^*) \, \mathbf{m} + \mathbf{a}_i^* \int_{\mathcal{D}_i} P_i(x|\mathbf{f}_i^*) \, \mathbf{m} \\ &\quad \int_{\mathcal{D}_i} P_i(x|\mathbf{f}_i^*) \, \mathbf{m} \quad r_i \geq b_i \|m_i^* - m_j^*\|, \end{aligned}$$

$$\int_{\mathcal{D}_i} P_i(x|\mathbf{f}_i^*) \, \mathbf{m} \leq \int_{\|x - m_i^*\| \leq b_i \|m_i^* - m_j^*\|} P_i(x|\mathbf{f}_i^*) \, \mathbf{m}.$$

$$\mathbf{B} \quad y = (x - m_i^*) / \|m_i^* - m_j^*\|,$$

$$\begin{aligned} &\int_{\mathcal{D}_i} P_i(x|\mathbf{f}_i^*) \, \mathbf{m} \\ &\leq \int_{\|y\| \leq b_i} w(\|m_i^* - m_j^*\| y + m_i^*) (1^i)^{-c_1} \exp(-\mathbf{r}(\|m_i^* - m_j^*\|^{c_2}) / (1^i)^{\mathbf{a}c_2} \|y\|^{c_2}) \|m_i^* - m_j^*\| \, \mathbf{m}' \\ &= \int_{\|y\| \leq b_i} \|m_i^* - m_j^*\| w(\|m_i^* - m_j^*\| y + m_i^*) (1^i)^{-c_1} \\ &\quad \times \exp(-\mathbf{r}(\|m_i^* - m_j^*\|^{c_2}) / (1^i)^{\mathbf{a}c_2} \|y\|^{c_2}) \, \mathbf{m}', \end{aligned} \quad (51)$$

$\mathbf{m}'$

$\mathbf{m}$

$$m_i^*$$

$$\frac{w(\|m_i^* - m_j^*\| y + m_i^*)}{\|m_i^* - m_j^*\|},$$

$$\|m_i^* - m_j^*\|^{-q} w(\|m_i^* - m_j^*\| y + m_i^*)$$

$$\mathbf{Z}(\mathbf{F}^*) \rightarrow 0.$$

$$\|m_i^* - m_j^*\|^{-q} w(\|m_i^* - m_j^*\|y + m_i^*) \leq M, \quad L \geq 1,$$

$$\|m_i^* - m_j^*\|^{1+q} (1^i)^{-c_1} \leq O(Z^{-c'_1}),$$

$$\|m_i^* - m_j^*\|^{c_2} (1^i)^{-nc_2} \geq O(Z^{-c_2}),$$

$$c'_1 = (q+1) \vee (c_1/n).$$

$$A \quad , \quad E \quad . \quad (51)$$

$$\begin{aligned} \int_{\mathcal{Q}_i} P_i(x|\mathbf{f}_i^*) \, \mathbf{m} &\leq \int_{\mathcal{B}_i} \frac{1}{Z^{c'_1}(\mathbf{F}^*)} w_1(y) \, -\mathbf{r}'(1/Z^{c_2}(\mathbf{F}^*))\|y\|^{c_2} \, \mathbf{m}' \\ &= \int_{\mathcal{B}_i} \frac{1}{Z^{c'_1}} w_1(y) \, -\mathbf{r}'(1/Z^{c_2})\|y\|^{c_2} \, \mathbf{m}', \end{aligned} \quad (52)$$

$$\mathcal{B}_i = \{y: \|y\| \geq b_i\}, \quad \mathbf{r}' \quad , \quad w_1(y)$$

$$F \quad ,$$

$$F_i(\mathbf{Z}) = \int_{\mathcal{B}_i} P(y|\mathbf{Z}) \, y, \quad P(y|\mathbf{Z}) = \frac{1}{Z^{c'_1}} w_1(y) \, -\mathbf{r}'(1/Z^{c_2})\|y\|^{c_2}$$

$$F_i(\mathbf{Z})/Z^p \quad \mathbf{Z} \quad .$$

$$F \quad y \in \mathcal{B}_i,$$

$$\begin{aligned} \lim_{Z \rightarrow 0} \frac{P(y|\mathbf{Z})}{Z^p} &= w_1(y) \lim_{Z \rightarrow 0} \frac{1}{Z^{(c'_1+p)}} \, -\mathbf{r}'(1/Z^{c_2})\|y\|^{c_2} \\ &= w_1(y) \lim_{Z=\frac{1}{Z} \rightarrow \infty} \frac{Z^{(c'_1+p)}}{Z^{c_2} \mathbf{r}'\|y\|^{c_2}} \\ &= 0, \end{aligned}$$

$$\mathcal{B}_i,$$

$$\begin{aligned} \lim_{Z \rightarrow 0} \frac{F_i(\mathbf{Z})}{Z^p} &= \lim_{Z \rightarrow 0} \int_{\mathcal{B}_i} \frac{P(y|\mathbf{Z})}{Z^p} \, \mathbf{m}' \\ &= \int_{\mathcal{B}_i} \lim_{Z \rightarrow 0} \frac{P(y|\mathbf{Z})}{Z^p} \, \mathbf{m}' \\ &= 0 \end{aligned}$$

$$F_i(\mathbf{Z}) = o(Z^p). \quad \mathbf{I} \quad E \quad . \quad (52)$$

$$\lim_{Z(\mathbf{F}^*)=Z} \int_{\mathcal{Q}_i} P_i(x|\mathbf{f}_i^*) \, \mathbf{m} = o(Z^p). \quad (53)$$

$$, \quad :$$

$$\lim_{Z(\mathbf{F}^*)=Z} \int_{\mathcal{Q}_j} P_j(x|\mathbf{f}_j^*) \, \mathbf{m} = o(Z^p).$$

A ,

$$\begin{aligned} f_{ij}(\mathbf{Z}) &= \frac{e_{ij}(\mathbf{F}^*)}{Z(\mathbf{F}^*)=Z} \\ &\leq \frac{Z(\mathbf{F}^*)=Z}{\left( \mathbf{a}_j^* \int_{\mathcal{D}_j} P_j(x|\mathbf{f}_j^*) \, \mathbf{m} + \mathbf{a}_i^* \int_{\mathcal{D}_i} P_i(x|\mathbf{f}_i^*) \, \mathbf{m} \right)} \\ &\leq \frac{Z(\mathbf{F}^*)=Z}{\int_{\mathcal{D}_j} P_j(x|\mathbf{f}_j^*) \, x + \int_{Z(\mathbf{F}^*)=Z} \int_{\mathcal{D}_i} P_i(x|\mathbf{f}_i^*) \, \mathbf{m}} \\ &= o(Z^p). \end{aligned}$$

$$f(\mathbf{Z}) \leq \sum_{ij} f_{ij}(\mathbf{Z}) = o(Z^p). \quad (54)$$

M ,

$$\begin{aligned} \frac{f^e(\mathbf{Z})}{Z^p} &= \lim_{Z \rightarrow 0} \left( \frac{f(\mathbf{Z})}{Z^e} \right)^e = 0, \\ f^e(\mathbf{Z}) &= o(Z^p) \quad f^e(\mathbf{Z}(\mathbf{F}^*)) = o(Z^p(\mathbf{F}^*)). \quad \square \end{aligned}$$

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