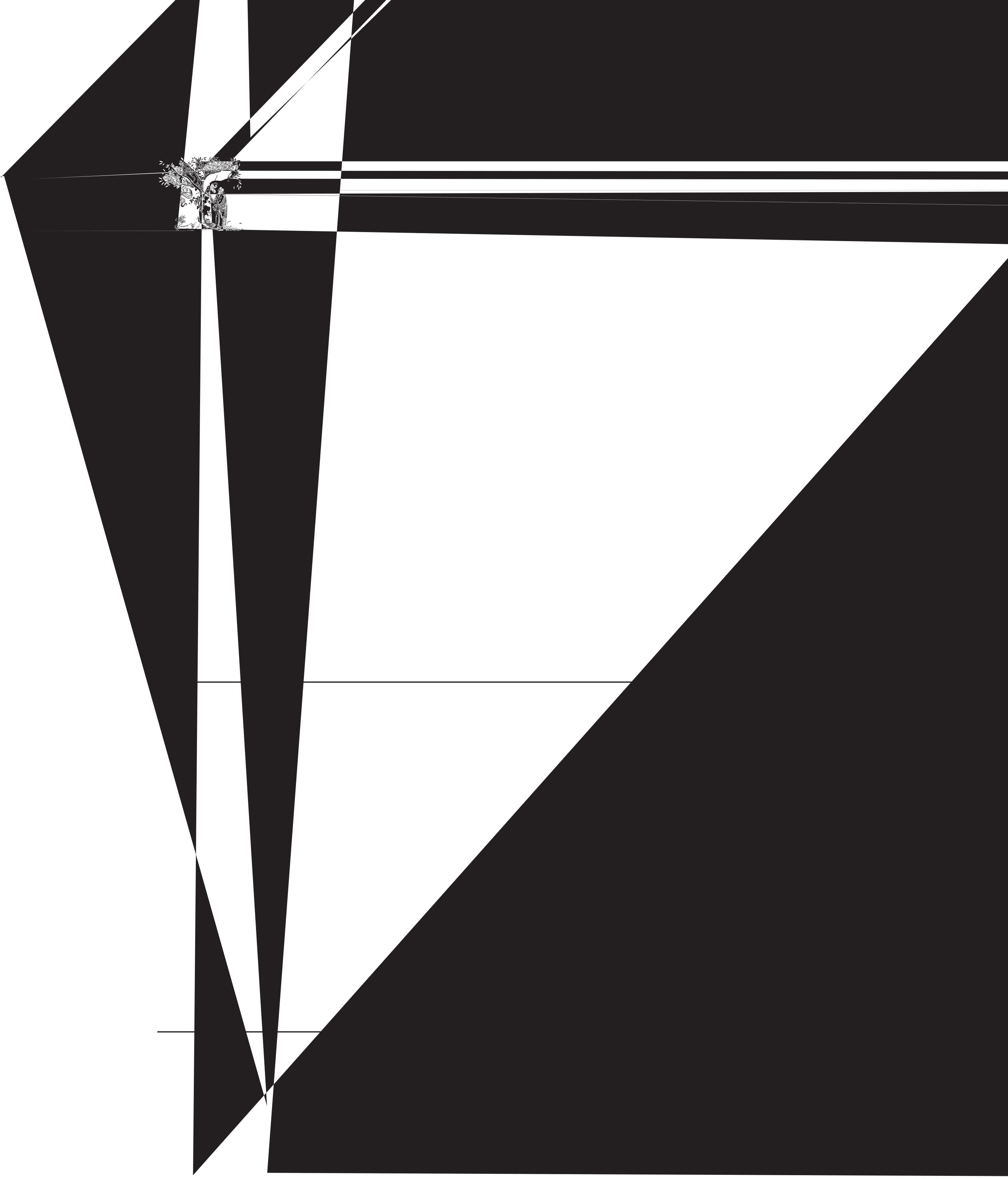

1. Introduction

EM (ML)
(MAP) [3]. EM
(. ., [2,4,12,15,16,18,19]). G ,
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[9], [5], -N
[6,10], [13] ” [7].
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2. The EM algorithm for mixtures of densities from exponential families

2.1. The mixture model

:

$$P(x|\mathrm{F}) = \sum_{i=1}^{K} \mathrm{a}_i P_i(x|\mathrm{f}_i), \quad \mathrm{a}_i \geqslant 0, \ \sum_{i=1}^{K} \mathrm{a}_i = 1, \tag{1}$$

$x = [x_1, \ldots, x_n] \ \in R^n$, P_i
$\mathrm{f}_i \in \mathrm{O}_i \subset R^{d_i}$, K
. F a_i
f_i, , $\mathrm{F} = (\mathrm{a}_1, \ldots, \mathrm{a}_K, \mathrm{f}_1, \ldots, \mathrm{f}_K) \in \mathrm{O}$,

$$\mathrm{O} = \left\{ (\mathrm{a}_1, \ldots, \mathrm{a}_K, \mathrm{f}_1, \ldots, \mathrm{f}_K) : \sum_{i=1}^{K} \mathrm{a}_i = 1 \quad \mathrm{a}_i \geqslant 0, \ \mathrm{f}_i \in \mathrm{O}_i \quad i = 1, \ldots, K \right\}.$$

I $P_i(x|\mathrm{f}_i) = P_i(x|m_i, \mathrm{S}_i)$ G

$$P_i(x|\mathrm{f}_i) = P(x|m_i, \mathrm{S}_i) = \frac{1}{(2\mathrm{p})^{n/2}(\quad \mathrm{S}_i)^{1/2}} \quad {}^{-(1/2)(x-m_i) \ \mathrm{S}_i^{-1}(x-m_i)}, \tag{2}$$

$m_i = [m_{i1}, \ldots, m_{in}]$, $\mathrm{S}_i = (\mathrm{s}^i_{kl})_{n \times n}$
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G
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G .
A $q(x|\mathrm{y}), \mathrm{y} \in \mathrm{Y} \subset R^d$ R^n

$$q(x|\mathrm{y}) = a(\mathrm{y})^{-1} b(x) \quad {}^{\mathrm{y} \ t(x)}, \quad x \in R^n, \tag{3}$$

$b(x), t(x)$ x R^n $a(\mathrm{y})$

$$a(\mathrm{y}) = \int_{R^n} b(x) \quad {}^{\mathrm{y} \ t(x)} \quad \mathrm{m}$$

m R^n. I $b(x) \geqslant 0$
$x \in R^n$, $a(\mathrm{y}) < +\infty$ $\mathrm{y} \in \mathrm{Y}$ $t(x)$, ,
. A ,
$b(x)$.

O ,

$$P(x|\mathrm{f}) = q(x|\mathrm{y}(\mathrm{f})) = a(\mathrm{f})^{-1}b(x) \quad ^{\mathrm{y}(\mathrm{f}) \;\; t(x)}, \quad x \in R^n$$

” $\mathrm{f} = E_{\mathrm{y}}(t(X))$

(1 18).

$P(x|\mathrm{f})$ m S,

$U(x|\mathrm{f})$:

$$P(x|\mathrm{f}) \leqslant U(x|\mathrm{f}) = w(x)(1 \quad)^{-c_1} \quad ^{-\mathrm{r}(1/\mathrm{Z}(x))^{-c_2}}, \tag{4}$$

$$\mathrm{Z}(x) = \frac{(1 \quad)^{\mathrm{n}}}{\|x - m\|}$$

l S $P(x|\mathrm{f})$. M , c_1, c_2, r n

, $w(x)$ $x_1, \ldots, x_n$

. H , E

. A ,

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G .

I , $P_i(x|\mathrm{f}_i)$ ” -

$\mathrm{f}_i \in \mathrm{O}_i \subset R^{d_i}$:

$$P_i(x|\mathrm{f}_i) = a_i(\mathrm{f}_i)^{-1}b_i(x) \quad ^{\mathrm{y}_i(\mathrm{f}_i) \;\; t_i(x)}, \quad x \in R^n \tag{5}$$

$\mathrm{F}^* = (\mathrm{a}_1^*, \ldots, \mathrm{a}_K^*, \mathrm{f}_1^*, \ldots, \mathrm{f}_K^*)$ ”

, ,

F^*. A , $t_i(x)$

$x_1, \ldots, x_n$.

M , $P_i(x|\mathrm{f}_i^*)$:

$$P_i(x|\mathrm{f}_i^*) \leqslant U_i(x|\mathrm{f}_i^*) = w(x)(1^i \quad)^{-c_1} \quad ^{-\mathrm{r}(1/\mathrm{Z}_i(x))^{c_2}}, \tag{6}$$

$$\mathrm{Z}_i(x) = \frac{(1^i \quad)^{\mathrm{n}_i}}{\|x - m_i^*\|}$$

m_i^* 1^i

S_i^* $P_i(x|\mathrm{f}_i^*)$, . c_1, c_2, r $w(x)$ i .

F , 1^i . n

n_i r $U_i(x|\mathrm{f}_i^*)$

. A ,

$$\mathrm{Z}_i(x) = \frac{(1^i \quad)^{\mathrm{n}}}{\|x - m_i^*\|}, \quad i = 1, \ldots, K,$$

n .

2.2. The EM algorithm and its asymptotic convergence rate

$\mathscr{S}_N = \{x^{(t)} : t = 1, \ldots, N\}$

f $_1, \ldots,$ f $_K$. I F $= ($a$_1, \ldots,$ a$_K,$ f$_1, \ldots,$ f$_K)$

ML - $L($F$) = \sum_{t=1}^{N}$ $P(x^{(t)}|$F$)$

E . (1), EM

$$\text{a}_i^+ = \frac{1}{N}\sum_{t=1}^{N} \frac{\text{a}_i P_i(x^{(t)}|\,\text{f}_i)}{P(x^{(t)}|\text{F}\ )}, \tag{7}$$

$$\text{f}_i^+ = \left\{\sum_{t=1}^{N} t_i(x^{(t)}) \frac{\text{a}_i P_i(x^{(t)}|\,\text{f}_i)}{P(x^{(t)}|\text{F}\ )}\right\} \Bigg/ \left\{\sum_{t=1}^{N} \frac{\text{a}_i P_i(x^{(t)}|\,\text{f}_i)}{P(x^{(t)}|\text{F}\ )}\right\}, \tag{8}$$

$i = 1, \ldots, K.$

$L($F$)$ [3,19] . M ,

, EM

$L($F$)$ [18] . I ,

EM F* (. .,

N $\mathscr{S}_N$, EM F^N

$_{N\to\infty}$F$^N =$ F*),

F*.

I [18] , EM

F$^+ = G($F $)$

$$\text{F}^+ - \text{F}^N = G(\text{F}\ ) - G(\text{F}^N) = G'(\text{F}^N)(\text{F}\ - \text{F}^N) + O(\|\text{F}\ - \text{F}^N\|^2) \tag{9}$$

F O F^N, $G'($F$)$ J $G($F$)$ F^N $O(x)$

$x \to 0$. B ,

N , , $G'($F$^N)$

$$E(G'(\text{F}^*)) = I - Q(\text{F}^*)R(\text{F}^*),$$

$$Q(\text{F}^*) = diag(\text{a}_1^*, \ldots, \text{a}_K^*, \text{a}_1^{*-1}\,\text{P}_1, \ldots, \text{a}_K^{*-1}\,\text{P}_K) \tag{10}$$

$$\text{P}_i = \int_{R^n} [t_i(x) - \text{f}_i^*][t_i(x) - \text{f}_i^*]\ P_i(x|\,\text{f}_i^*)\ \text{m}$$

$$R(\text{F}^*) = \int_{R^n} V(x)V(x)\ P(x|\text{F}^*)\ \text{m} \tag{11}$$

$$V(x) = (\text{b}_1(x), \ldots, \text{b}_K(x), \text{a}_1^*\text{b}_1(x)\text{G}_1(x)\ , \ldots, \text{a}_K^*\text{b}_K(x)\text{G}_K(x)\ )\ ,$$

$$\text{b}_i(x) = P_i(x|\,\text{f}_i^*)/P(x|\text{F}^*),$$

$$\text{G}_i(x) = \text{P}_i^{-1}[t_i(x) - \text{f}_i^*].$$

H , $E(\cdot) = E_{\Phi^*}(\cdot)$. I E . (9)
EM Φ^N $\|G'(\Phi^N)\|$. B
N ,
r EM Φ^*:

$$r \leqslant \lim_{N\to\infty} \|G'(\Phi^N)\| = \left\| \lim_{N\to\infty} G'(\Phi^N) \right\|$$
$$= \|E(G'(\Phi^*))\| = \|I - Q(\Phi^*)R(\Phi^*)\|. \tag{12}$$

I , EM

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3. The main result

3.1. The measures of the overlap

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G . E .
(1) Φ^*:

$$h_i(x) = \frac{\alpha_i^* P_i(x|\phi_i^*)}{\sum_{j=1}^K \alpha_j^* P_j(x|\phi_j^*)} \qquad i = 1, \ldots, K. \tag{13}$$

I E . (11)

$$h_i(x) = \alpha_i^* \beta_i(x). \tag{14}$$

$$\gamma_{ij}(x) = (\delta_{ij} - h_i(x))h_j(x) \qquad i,j = 1, \ldots, K, \tag{15}$$

δ_{ij} K . ,
:

$$e_{ij}(\Phi^*) = \int_{R^n} |\gamma_{ij}(x)| P(x|\Phi^*) \, \mathrm{d}\mu$$

$i,j = 1, 2, \ldots, K$, $e_{ij}(\Phi^*) \leqslant 1$ $|\gamma_{ij}(x)| \leqslant 1$.
F $i \neq j$, $e_{ij}(\Phi^*)$
i j . $P_i(x|\phi_i^*)$ $P_j(x|\phi_j^*)$
x, $h_i(x)h_j(x)$

$$e(F^*) = 0$$

$h_i(x)h_j(x) = 0$ $i \neq j$

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$e(F^*)$. A
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,
$e(F^*)$.

3.2. Regular conditions and lemmas

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(1) *Nondegenerate condition on the mixing proportions*:
:

$$a_i^* \geqslant a \qquad i = 1, \ldots, K, \tag{16}$$

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(2) *Uniform attenuating condition on the eigenvalues of the covariance matrices*:
S_i^* i $l_{i1}, \ldots, l_{in}$
.

$$b\, l(F^*) \leqslant l_{ij} \leqslant l(F^*) \qquad i = 1, \ldots, K,\ k = 1, \ldots, n, \tag{17}$$

b $l(F^*)$
$S_1^*, \ldots, S_K^*$, . .,

$$l(F^*) = \quad_{i,j} \quad l_{ij}$$

B. ,
. I
E . (17) K
, . .,

$$1 \leqslant k(S_i^*) \leqslant B' \qquad i = 1, \ldots, K,$$

$k(S_i^*)$ S_i^* B' .

(3) *Regular condition on the mean vectors*:

, . ., $m_1^*, \ldots, m_K^*$,

$$\mathrm{m}D \quad (\mathrm{F}^*) \leqslant D \quad (\mathrm{F}^*) \leqslant \|m_i^* - m_j^*\| \leqslant D \quad (\mathrm{F}^*) \qquad i \neq j, \tag{18}$$

$D \quad (\mathrm{F}^*) = \quad {}_{i\neq j}\|m_i^* - m_j^*\|$, $D \quad (\mathrm{F}^*) = \quad {}_{i\neq j}\|m_i^* - m_j^*\|$, m

. ,

. M ,

, m_i^*, m_j^* , . .,

T $\|m_i^* - m_j^*\| \geqslant T$ $i \neq j$. I ,

E . (18) $m_1^*, \ldots, m_K^*$.

$\| \mathrm{P}_i^{-1} \|$, P_i E . (10).

$P_i(x | \mathrm{f}_i^*)$,

11 :

$\mathrm{P}_i =$

$Z(F^*) \to 0.$, $Z(F^*) \to 0$ $e(F^*) \to 0$ $Z(F^*) \to 0$ $Z(F^*) \to 0$.
A
, $e(F^*)$
, $Z(F^*) \to 0$ $e(F^*) \to 0$.
$Z(F^*)$, . ., F^*
$Z(F^*)$,

$$f(Z) = \quad_{Z(F^*)=Z} e(F^*) \tag{21}$$

$e(F^*)$ 1. B ,

$$e_{ij}(F^*) \leqslant e(F^*) \leqslant f(Z(F^*)) \qquad i \neq j. \tag{22}$$

F , (A A).

Lemma 1. *Suppose that a mixture of K densities from the bell sheltered exponential families of the parameter* F^* *satisfies Conditions* (1) (3). *As* $Z(F^*)$ *tends to zero*, *we have*

( ) $Z(F^*), Z_i(m_j^*)$ *and* $Z_j(m_i^*)$ *are the equivalent infinitesimals.*
( ) *For* $i \neq j$, *we have*

$$\|m_i^*\| \leqslant T' \|m_i^* - m_j^*\|, \tag{23}$$

where T' *is a positive number.*
( ) *For any two nonnegative numbers with* $p + q > 0$, *we have*

$$\|m_i^* - m_j^*\|^p (1^i \quad)^{-nq} \leqslant O(Z^{-p \vee q}(F^*)), \tag{24}$$

where $p \vee q = \quad \{p, q\}$.

Lemma 2. *Suppose that a mixture of K densities from the bell sheltered exponential families of the parameter* F^* *satisfies Conditions* (1) (3). *As* $Z(F^*)$ *tends to zero*, *we have for each i*

( ) $$\| P_i \| \leqslant c \|m_i^* - m_j^*\|^p, \tag{25}$$

where $j \neq i$, c *and p are some positive numbers.*

( ) $$E(\|t_i(X) - f_i^*\|^2) \leqslant u M_i^q(F^*), \tag{26}$$

where $M_i(F^*) = \quad_{j \neq i} \|m_i^* - m_j^*\|$, u *and q are some positive numbers.*

Lemma 3. *Suppose that a mixture of K densities from the bell sheltered exponential families of the parameter* F^* *satisfies Conditions* (1) (3) *and* $Z(F^*) \to 0$ *as an infinitesimal*, *we have*

$$f^e(Z(F^*)) = o(Z^p(F^*)), \tag{27}$$

where $e > 0$, p *is any positive number and* $o(x)$ *means that it is a higher order infinitesimal as* $x \to 0$.

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$e(F^*)$ $Z(F^*)$ L 3
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3.3. The main theorem

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Theorem 1. *Given a mixture of K densities from the bell sheltered exponential families of the parameter* F^* *that satisfies Conditions* (1) (4), *as* $e(F^*)$ *tends to zero as an infinitesimal, we have*

$$r \leqslant \|E(G'(F^*))\| = o(^{0.5-e}(F^*)), \tag{28}$$

where e *is an arbitrarily small positive number.*

A , ,
, , $e(F^*) \to 0$, $\|E(G'(F^*))\|$
$^{0.5-e}(F^*)$. , (F^*) ,
EM F^*
$^{0.5-e}(F^*)$. , $e(F^*)$ N ,
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Proof of Theorem 1. $Q(F^*)R(F^*)$.
A $Q(F^*)$ $R(F^*)$,
$Q(F^*)R(F^*)$, :

$$Q(F^*)R(F^*) = diag[diag[\mathscr{A}\ ], a^{*-1} P_1, \ldots, a_K^{*-1} P_K] \times \begin{pmatrix} R_{b,b} & R_{b,G_1} & \cdots & R_{b,G_K} \\ R_{G_1,b} & R_{G_1,G_1} & \cdots & R_{G_1,G_K} \\ \vdots & \vdots & \ddots & \vdots \\ R_{G_K,b} & R_{G_K,G_1} & \cdots & R_{G_K,G_K} \end{pmatrix}$$

$$= \begin{pmatrix} diag[\mathscr{A}\ ]R_{b,b} & diag[\mathscr{A}\ ]R_{b,G_1} & \cdots & diag[\mathscr{A}\ ]R_{b,G_K} \\ a_1^{*-1} P_1 R_{G_1,b} & a_1^{*-1} P_1 R_{G_1,G_1} & \cdots & a_1^{*-1} P_1 R_{G_1,G_K} \\ \vdots & \vdots & \ddots & \vdots \\ a_K^{*-1} P_K R_{G_K,b} & a_K^{*-1} P_K R_{G_K,G_1} & \cdots & a_K^{*-1} P_K R_{G_K,G_K} \end{pmatrix},$$

$b(x) = [b_1(x), \ldots, b_K(x)]$ $\mathscr{A} = [a_1^*, \ldots, a_K^*]$.
$R(F^*)$ $V(x)$

$$V(x) = [b(x)\ , a_1^* b_1(x) G_1(x)\ , \ldots, a_K^* b_K(x) G_K(x)\ ]\ .$$

( ) *The computation of* $diag[\mathscr{A}\ ]R_{b,b}$: F $b_i(x)$
$h_i(x) = a_i^* b_i(x)$,

$$\int_{R^n} b_i(x) b_j(x) P(x|F^*)\ \ m = \frac{1}{a_i^* a_j^*} e_{ij}(F^*) \qquad i \neq j,$$
$$\int_{R^n} b_i^2(x) P(x|F^*)\ \ m = \frac{1}{a_i^*} - \frac{1}{(a_i^*)^2} e_{ii}(F^*)$$

$$diag[\mathscr{A}\ ]R_{b,b}\ = I_K + \begin{pmatrix} -a_1^{*-1} e_{11}(F^*) & a_2^{*-1} e_{12}(F^*) & \cdots & a_K^{*-1} e_{1K}(F^*) \\ a_1^{*-1} e_{21}(F^*) & -a_2^{*-1} e_{22}(F^*) & \cdots & a_K^{*-1} e_{2K}(F^*) \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{*-1} e_{K1}(F^*) & a_2^{*-1} e_{K2}(F^*) & \cdots & -a_K^{*-1} e_{KK}(F^*) \end{pmatrix}.$$

B

$$\frac{1}{a_j^*} e_{ij}(F^*) \leqslant \frac{1}{a} e_{ij}(F^*) = o(\ ^{0.5-e}(F^*)),$$

$$diag[\mathscr{A}\ ]R_{\mathrm{b,b}} = I_K + o$$

, . I
C $|\mathrm{g}_{ij}(x)|\leqslant 1$

$$
\begin{aligned}
&|E(h_j(X)(h_i(X)-\mathrm{d}_{ij})(t_{i,k}(X)-\mathrm{f}^*_{i,k}))|\\
&\quad\leqslant E(|h_j(X)(h_i(X)-\mathrm{d}_{ij})||(t_{i,k}(X)-\mathrm{f}^*_{i,k})|)\\
&\quad\leqslant E^{1/2}(\mathrm{g}^2_{ij}(X))E^{1/2}((t_{i,k}(X)-\mathrm{f}^*_{i,k})^2)\\
&\quad\leqslant E^{1/2}(|\mathrm{g}_{ij}(X))E^{1/2}((t_{i,k}(X)-\mathrm{f}^*_{i,k})^2)\\
&\quad\leqslant \sqrt{e_{ij}(\mathrm{F}^*)}E^{1/2}((t_{i,k}(X)-\mathrm{f}^*_{i,k})^2).
\end{aligned}
$$

A L 2, $E(\|t_i(X)-\mathrm{f}^*_i\|^2|\mathrm{F}^*)$ $\mathrm{u}M^q_i(\mathrm{F}^*)$. ,
$E^{1/2}((t_{i,k}(X)-\mathrm{f}^*_{i,k})^2)$ $\sqrt{\mathrm{u}M^q_i(\mathrm{F}^*)}$.
,

$$E(diag[\mathscr{A}\ ]\mathrm{a}^*_i\mathrm{b}_i(X)\mathrm{b}(X)(t_i(X)-\mathrm{f}^*_i)\ )=O(M^{q/2}_i(\mathrm{F}^*)e^{0.5}(\mathrm{F}^*)).$$

A L 1 3, $M^{q/2}_i(\mathrm{F}^*)\ ^{0.5}(\mathrm{F}^*)$ $e(\mathrm{F}^*)$
$\mathrm{Z}(\mathrm{F}^*)$. I

$$\|E(diag[\mathscr{A}\ ]\mathrm{a}^*_i\mathrm{b}_i(X)\mathrm{b}(X)(t_i(X)-\mathrm{f}^*_i)\ )\|=O(M^{q/2}_i(\mathrm{F}^*)\ ^{0.5}(\mathrm{F}^*)).$$

M ,

$$\|diag[\mathscr{A}\ ]R_{\mathrm{b},\mathrm{G}_i}\|\leqslant\|E(diag[\mathscr{A}\ ]\mathrm{a}^*_i\mathrm{b}_i(X)\mathrm{b}(X)(t_i(X)-\mathrm{f}^*_i)\ )\|\|\mathrm{P}^{-1}_i\|$$

$$\|\mathrm{P}^{-1}_i\|=\|I(\mathrm{f}^*_i)\|\leqslant O(\|m^*_i\|^{\mathrm{t}_1}(1^i\quad)^{-\mathrm{t}_2})$$

C (4). ,

$$\|diag[\mathscr{A}\ ]R_{\mathrm{b},\mathrm{G}_i}\|\leqslant\mathrm{u}\|m^*_i-m^*_{j'}\|^{q_1}(1^i\quad)^{-q_2}\ ^{0.5}(\mathrm{F}^*),$$

$q_1=(q/2)+\mathrm{t}_1$, $q_2=\mathrm{t}_2$, u .
, L 1 3

$$\|diag[\mathscr{A}\ ]R_{\mathrm{b},\mathrm{G}_i}\|\leqslant O(\mathrm{Z}^{-q_1\vee q_2}(\mathrm{F}^*))e^{0.5}(\mathrm{F}^*)=o(e^{0.5-\mathrm{e}}(\mathrm{F}^*)).$$

B ,

$$diag[\mathscr{A}\ ]R_{\mathrm{b},\mathrm{G}_i}=o(\ ^{0.5-\mathrm{e}}(\mathrm{F}^*)).$$

( ) *The computation of* $\mathrm{a}^{*-1}_i\mathrm{P}_iR_{\mathrm{G}_i,\mathrm{b}}$ $(i=1,\ldots,K)$: A (1)
L 2, $\mathrm{a}^{*-1}_i\|\mathrm{P}_i\|$ $(1/\mathrm{a})\mathrm{c}\|m^*_i-m^*_j\|^p$, $j\neq i$, c p
. B $R_{\mathrm{G}_i,\mathrm{b}}=R_{\mathrm{b},\mathrm{G}_i}$, ( ) :

$$\mathrm{a}^{*-1}_i\mathrm{P}_iR_{\mathrm{G}_i,\mathrm{b}}\ =o(\ ^{0.5-\mathrm{e}}(\mathrm{F}^*)).$$

( ) *The computation of* $\mathrm{a}_i^{*-1}\,\mathrm{P}_i R_{\mathrm{G}_i,\mathrm{G}_i}\ (i=1,\ldots,K)$: B $V(x)$,

$$\begin{aligned}
\mathrm{a}_i^{*-1}\,\mathrm{P}_i R_{\mathrm{G}_i,\mathrm{G}_i} &= \mathrm{a}_i^{*-1}\,\mathrm{P}_i E(h_i^2(X)\mathrm{G}_i(X)\mathrm{G}_i\ (X)) \\
&= \mathrm{a}_i^{*-1} E(h_i^2(X)(t_i(X)-\mathrm{f}_i^*)(t_i(X)-\mathrm{f}_i^*)\ )\,\mathrm{P}_i^{-1} \\
&= I_{d_i} + \mathrm{a}_i^{*-1} E(h_i(X)(h_i(X)-1)(t_i(X)-\mathrm{f}_i^*)(t_i(X)-\mathrm{f}_i^*)\ )\,\mathrm{P}_i^{-1},
\end{aligned}$$

:

$$\mathrm{P}_i E(h_i(X)\mathrm{G}_i(X)\mathrm{G}_i\ (X)) = \mathrm{a}_i^* I_{d_i}.$$

F , $E(\|t_i(X)-\mathrm{f}_i^*\|^2|\mathrm{F}^*)$ $\mathrm{u}M_i^q(\mathrm{F}^*)$ a_i^{*-1} , , :

$$\mathrm{a}_i^{*-1} E(h_i(X)(h_i(X)-1)(t_i(X)-\mathrm{f}_i^*)(t_i(X)-\mathrm{f}_i^*)\ )\,\mathrm{P}_i^{-1} = o(\ ^{0.5-\mathrm{e}}(\mathrm{F}^*))$$

,

$$\mathrm{a}_i^{*-1}\,\mathrm{P}_i R_{\mathrm{G}_i,\mathrm{G}_i} = I_{d_i} + o(\ ^{0.5-\mathrm{e}}(\mathrm{F}^*)).$$

( ) *The computation of* $\mathrm{a}_i^{*-1}\,\mathrm{P}_i R_{\mathrm{G}_i,\mathrm{G}_j}\ (j\neq i)$: B $V(x)$,

$$\begin{aligned}
\mathrm{a}_i^{*-1}\,\mathrm{P}_i R_{\mathrm{G}_i,\mathrm{G}_j} &= \mathrm{a}_i^{*-1} E(\mathrm{a}_i^*\mathrm{b}_i(X)\mathrm{a}_j^*\mathrm{b}_j(X)(t_i(X)-\mathrm{f}_i^*)(t_j(X)-\mathrm{f}_j^*)\ )\,\mathrm{P}_j^{-1} \\
&= \mathrm{a}_i^{*-1} E(h_i(X)h_j(X)(t_i(X)-\mathrm{f}_i^*)(t_j(X)-\mathrm{f}_j^*)\ )\,\mathrm{P}_j^{-1}.
\end{aligned}$$

( ),

$$\mathrm{a}_i^{*-1}\,\mathrm{P}_i R_{\mathrm{G}_i,\mathrm{G}_j} = o(\ ^{0.5-\mathrm{e}}(\mathrm{F}^*)).$$

( ) ( ), :

$$Q(\mathrm{F}^*)R(\mathrm{F}^*) = I + o(\ ^{0.5-\mathrm{e}}).$$

, E . (12),

$$r \leqslant \|I - Q(\mathrm{F}^*)R(\mathrm{F}^*)\| = o(\ ^{0.5-\mathrm{e}}(\mathrm{F}^*)). \quad \square$$

4. A typical class: Gaussian mixtures

EM G . A 1 , G $P_i(x|m_i,\mathrm{S}_i)$ E . (2) $\mathrm{y}_i = (\mathrm{S}_i^{-1}m_i, \mathrm{S}_i^{-1})$ $t_i(x) = (x, -\frac{1}{2}xx\ )$. , f_i, y_i, $(m_i, -\frac{1}{2}(\mathrm{S}_i + m_i m_i\ ))$ (m_i, S_i) .

N , G

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.

Lemma 4. *Suppose that* $P_i(x|\,\mathrm{f}_i^*) = P_i(x|m_i^*, \mathrm{S}_i^*)$ *is a Gaussian distribution with the mean* m_i^* *and the covariance matrix* S_i^*, *and that the condition number of* S_i^*, *i.e.*, $\mathrm{k}(\mathrm{S}_i^*)$, *is upper bounded by* B'. *We have that* $P_i(x|\,\hat{\mathrm{f}}_i^*)$ *is bell-sheltered, i.e.*,

$$P_i(x|\,\mathrm{f}_i^*) = P_i(x|m_i^*, \mathrm{S}_i^*) \leqslant b\frac{1}{(1^i \quad)^{n/2}} \quad ^{-(1/21^i \quad)\|x-m_i^*\|^2}, \tag{29}$$

where b is a positive number.

Proof. B $y = U_i(x - m_i^*)$

$$P(y|1^i \quad) = \frac{1}{(2\mathrm{p}1^i \quad)^{n/2}} \quad ^{-(1/21^i \quad)\|y\|^2},$$

$$P_i(x|m_i^*, \mathrm{S}_i^*) \leqslant B'^{n/2} P(y|1^i \quad),$$

$\mathrm{k}(\mathrm{S}_i^*) \leqslant B'$. M , $\|y\| = \|x - m_i^*\|$,

$$P_i(x|m_i^*, \mathrm{S}_i^*) \leqslant b\frac{1}{(1^i \quad)^{n/2}} \quad ^{-(1/21^i \quad)\|x-m_i^*\|^2},$$

$b = (B'/2\mathrm{p})^{n/2}$. □

B L 4, (1) (3), G F^*
K .
M , $P_i(x|\,\mathrm{f}_i^*) = P_i(x|m_i^*, \mathrm{S}_i^*)$, $t_i(x)$
:

$$t_i(x) = \begin{cases} x & m_i^*, \\ -\frac{1}{2}xx & -\frac{1}{2}(\mathrm{S}_i^* + m_i^*(m_i^*) \quad). \end{cases}$$

$t_i(x)$ $x_1, \ldots, x_n$.
, G F^* (1) (3)
(4).
F , G , (4) (2),
. F ,
$\mathrm{f}_i^* = [(m_i^*) \quad , vec[\dot{\mathrm{S}}_i^*] \quad] \quad$, $\dot{\mathrm{S}}_i^* = -\frac{1}{2}(\mathrm{S}_i^* + m_i^*(m_i^*) \quad)$, $\hat{\mathrm{f}}_i^* = [(m_i^*) \quad , vec[\mathrm{S}_i^*] \quad] \quad$.

Lemma 5. *Suppose that* $P_i(x|\,\mathrm{f}_i^*) = P_i(x|m_i^*, \mathrm{S}_i^*)$ *is a Gaussian density and* $\mathrm{k}(\mathrm{S}_i^*)$ *is upper bounded. As* 1^i *tends to zero, we have*

$$\|I(\mathrm{f}_i^*)\| = O((1^i \quad)^{-\mathrm{t}}), \tag{30}$$

where t *is a positive number.*

Proof. B ,

$$\frac{\partial P_i(x|m_i^*, \mathrm{S}_i^*)}{\partial m_i^*} = (x - m_i^*)\mathrm{S}_i^* P_i(x|m_i^*, \mathrm{S}_i^*), \quad (31)$$

$$\frac{\partial P_i(x|m_i^*, \mathrm{S}_i^*)}{\partial \mathrm{S}_i^*} = -\frac{1}{2}(\mathrm{S}_i^{*-1} - \mathrm{S}_i^{*-1}(x - m_i^*)(x - m_i^*) \ \mathrm{S}_i^{*-1})P_i(x|m_i^*, \mathrm{S}_i^*). \quad (32)$$

A F ,

$$\begin{aligned} I(\mathrm{f}_i^*) &= E_{\mathrm{f}_i^*}\left(\left(\frac{\partial P_i(X|\,\mathrm{f}_i^*)}{\partial\,\mathrm{f}_i^*}\right)\left(\frac{\partial P_i(X|\,\mathrm{f}_i^*)}{\partial\,\mathrm{f}_i^*}\right) \ \right) \\ &= E_{\mathrm{f}_i^*}\left(\frac{\partial(\hat{\mathrm{f}}_i^*)}{\partial\,\mathrm{f}_i^*} \left(\frac{\partial P_i(X|\,\mathrm{f}_i^*)}{\partial\,\hat{\mathrm{f}}_i^*}\right)\left(\frac{\partial P_i(X|\,\mathrm{f}_i^*)}{\partial\,\hat{\mathrm{f}}_i^*}\right) \ \left(\frac{\partial(\hat{\mathrm{f}}_i^*)}{\partial\,\mathrm{f}_i^*}\right) \ \right) \\ &= \frac{\partial(\hat{\mathrm{f}}_i^*)}{\partial\,\mathrm{f}_i^*} I(\hat{\mathrm{f}}_i^*)\left(\frac{\partial(\hat{\mathrm{f}}_i^*)}{\partial\,\mathrm{f}_i^*}\right) \ , \end{aligned}$$

$$I(\hat{\mathrm{f}}_i^*) = E_{\mathrm{f}_i^*}\left(\left(\frac{\partial P_i(X|\,\mathrm{f}_i^*)}{\partial\,\hat{\mathrm{f}}_i^*}\right)\left(\frac{\partial P_i(X|\,\mathrm{f}_i^*)}{\partial\,\hat{\mathrm{f}}_i^*}\right) \ \right).$$

I E . (31) (32)
$P_i^3(x|m_i^*, \mathrm{S}_i^*)$ $I(\hat{\mathrm{f}}_i^*)$ G
$P_i(x|m_i^*, \frac{1}{3}\mathrm{S}_i^*)$ $|\mathrm{S}_i^*|$
,

$$I(\hat{\mathrm{f}}_i^*) = E_{(m_i^*,(1/3)\mathrm{S}_i^*)}(G(X, \mathrm{f}_i^*)),$$

$G(x, \mathrm{f}_i^*)$ $x - m_i^*$ S_i^*.
$y = x - m_i^*$,

$$I(\hat{\mathrm{f}}_i^*) = E_{(0,(1/3)\mathrm{S}_i^*)}(G(Y, \mathrm{S}_i^*)),$$

$G(y, \mathrm{S}_i^*)$ $g_{pq}(y, \mathrm{S}_i^*)$
$y_1, \ldots, y_n$. I S_i^{*-1}

$$\mathrm{S}_i^{*-1} = |\mathrm{S}_i^*|^{-1}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d_i} \\ a_{21} & a_{22} & \cdots & a_{2d_i} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d_i1} & a_{d_i2} & \cdots & a_{d_id_i} \end{pmatrix},$$

a_{kl} s_{kl}^{*i} S_i^*,
$g_{pq}(y, \mathrm{S}_i^*)$ s_{kl}^{*j}
$|\mathrm{S}_i^*|$. s_{kl}^{*i}

, t
$(1^i \quad)^{\mathrm{t}} g_{pq}(y, \mathrm{S}_i^*)$. B
G $1 \quad < B,$ $E_{(0,1/3\mathrm{S}_i^*)}((1^i \quad)^{\mathrm{t}} g_{pq}(Y, \mathrm{S}_i^*))$
. ,

$$\begin{aligned} \|I(\hat{\mathrm{f}}_i^*)\| &= \|(1^i \quad)^{-\mathrm{t}}(1^i \quad)^{\mathrm{t}} I(\mathrm{f}_i^*)\| \\ &= (1^i \quad)^{-\mathrm{t}} \|(1^i \quad)^{\mathrm{t}} I(\mathrm{f}_i^*)\| \\ &\leqslant \mathrm{o}(1^i \quad)^{-\mathrm{t}}, \end{aligned}$$

o . B

$$\|I(\mathrm{f}_i^*)\| \leqslant \left\| \frac{\partial(\hat{\mathrm{f}}_i^*)}{\partial \mathrm{f}_i^*} \right\| \|I(\hat{\mathrm{f}}_i^*)\| \left\| \left(\frac{\partial(\hat{\mathrm{f}}_i^*)}{\partial \mathrm{f}_i^*} \right) \quad \right\| = 4\|I(\hat{\mathrm{f}}_i^*)\|,$$

$\|\partial(\hat{\mathrm{f}}_i^*) \; / \partial \mathrm{f}_i^*\| = \|(\partial(\hat{\mathrm{f}}_i^*) \; / \partial \mathrm{f}_i^*) \; \| = 2,$
, E . (30) . □

, (1) (3)
EM G
, ,

Theorem 2. *Given a Gaussian mixture of K densities of the parameter* F^* *that satisfies conditions* (1) (3), *as* $e(\mathrm{F}^*)$ *tends to zero as an infinitesimal*, *we have*

$$\|G'(\mathrm{F}^*)\| = o(\;^{0.5-\mathrm{e}}(\mathrm{F}^*)), \tag{33}$$

where e *is an arbitrarily small positive number*.

I , 1 G
(1) (3) .

5. Conclusions

I ,
, EM
-N . M ,
.

Acknowledgements

G G C
H K A (P . C HK4225/04E) G N
F C (P . 60071004).

Appendix

Proof of Lemma 1. ( ). F ,
$Z(F^*) = \quad _{i \neq j} Z_i(m_j^*) = Z_{i'}(m_{j'}^*)$. A (2) (3),
$a_1, a_2, b_1, b_2, c_1, c_2$

$$a_1(1^{i'} \quad)^{n} \leqslant (1^{i} \quad)^{n} \leqslant a_2(1^{i'} \quad)^{n}, \tag{34}$$

$$b_1(1^{i} \quad)^{n} \leqslant (1^{j} \quad)^{n} \leqslant b_2(1^{i} \quad)^{n}, \tag{35}$$

$$c_1 \|m_{i'}^* - m_{j'}^*\| \leqslant \|m_i^* - m_j^*\| \leqslant c_2 \|m_{i'}^* - m_{j'}^*\|. \tag{36}$$

C E . (34) (35) E . (36),
a_1', a_2', b_1', b_2'

$$a_1' Z(F^*) \leqslant Z_i(m_j^*) \leqslant a_2' Z(F^*),$$

$$b_1' Z_i(m_j^*) \leqslant Z_j(m_i^*) \leqslant b_2' Z_i(m_j^*).$$

, $Z(F^*), Z_i(m_j^*)$ $Z_j(m_i^*)$.
( ), $\|m_i^* - m_j^*\|$ $Z(F^*) \to 0$, E . (23)
. I $\|m_i^* - m_j^*\|$ $Z(F^*) \to 0$,
$\|m_i^*\| \leqslant \|m_i^* - m_j^*\| + \|m_j^*\|$, E . (23) $\|m_i^*\|$ $\|m_j^*\|$
. O , $\|m_i^*\|$ $\|m_j^*\|$,
$\|m_i^*\|$ $\|m_i^* - m_j^*\|$,
E . (23). , ( ) .
F , ( ) .
I $p = q > 0$, ( ),

$$\|m_i^* - m_j^*\|^p (1^i \quad)^{-nq} = \|m_i^* - m_j^*\|^p (1^i \quad)^{-np}$$
$$= (Z_i(m_j^*))^{-p} = O(Z^{-p}(F^*)) = O(Z^{-p \vee q}(F^*)).$$

I $p > q$, 1^i ( ),

$$\|m_i^* - m_j^*\|^p (1^i \quad)^{-nq} \leqslant O(Z^{-p}(F^*)) = O(Z^{-p \vee q}(F^*)).$$

I $p < q$, $\|m_i^* - m_j^*\| \geqslant T$,

$$\|m_i^* - m_j^*\|^p (1^i \quad)^{-nq} \leqslant O(Z^{-q}(F^*)) = O(Z^{-p \vee q}(F^*)).$$

,

$$\|m_i^* - m_j^*\|^p (1^i \quad)^{-nq} \leqslant O(Z^{-p \vee q}(F^*)). \qquad \square$$

Proof of Lemma 2. ( ). A ,

$$\| P_i \| = \| E_{f_i^*}((t_i(X) - f_i^*)(t_i(X) - f_i^*) \quad) \| \leqslant E_{f_i^*}(\| t_i(X) - f_i^* \|^2). \tag{37}$$

$t_i(x)$ $x_1, x_2, \ldots, x_n$, . ., x,
:

$$t_i(x) = P_0 + P_1 x + P_2 x^2 + \cdots + P_k x^k,$$

$k \geqslant 0$, P_i $d_i \times n^i$, x^i
$x_{j_1} x_{j_2} \cdots x_{j_i}$, x_{j_p}
$x_1, x_2, \ldots, x_n$. I $\|x^i\| \leqslant \sqrt{n}\|x\|^i$ $i = 0, 1, \ldots, k$.
B ,

$$\begin{aligned} t_i(x) &= t_i(x - m_i^* + m_i^*) \\ &= P'_0 + P'_1(x - m_i^*) + P'_2(x - m_i^*)^2 + \cdots + P'_k(x - m_i^*)^k, \end{aligned} \tag{38}$$

P'_i $d_i \times n^i$, $m_{i1}^*, \ldots, m_{in}^*$.

$$\mathrm{f}_i^* = E_{\mathrm{f}_i^*}(t_i(X)) = P'_0 + E_{\mathrm{f}_i^*}(P'_1(X - m_i^*)) + \cdots + E_{\mathrm{f}_i^*}(P'_k(X - m^*)^k) \tag{39}$$

$$E_{\mathrm{f}_i^*}(P'_1(X - m_i^*)) = P'_1 E_{\mathrm{f}_i^*}(X - m_i^*) = 0,$$

$$t_i(X) - \mathrm{f}_i^* = \sum_{j=1}^{k} [P'_j(X - m_i^*)^j - E_{\mathrm{f}_i^*}(P'_j(X - m_i^*)^j)].$$

N ,

$$\begin{aligned} E_{\mathrm{f}_i^*}(\|t_i(X) - \mathrm{f}_i^*\|^2) &= E_{\mathrm{f}_i^*}(\|(t_i(X) - \mathrm{f}_i^*)\ \ (t_i(X) - \mathrm{f}_i^*)\|) \\ &= E_{\mathrm{f}_i^*}\left(\left\| \sum_{j_1=1, j_2=1}^{k} [P'_{j_1}(X - m_i^*)^{j_1} - E_{\mathrm{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})] \right.\right. \\ &\quad \left.\left. \times [P'_{j_2}(X - m_i^*)^{j_2} - E_{\mathrm{f}_i^*}(P'_{j_2}(X - m_i^*)^{j_2})] \right\|\right) \\ &\leqslant \sum_{j_1=1, j_2=1}^{k} E_{\mathrm{f}_i^*}(\|[P'_{j_1}(X - m_i^*)^{j_1} - E(P'_{j_1}(X - m_i^*)^{j_1})]\ \ \| \\ &\quad \times \|[P'_{j_2}(X - m_i^*)^{j_2} - E_{\mathrm{f}_i^*}(P'_{j_2}(X - m_i^*)^{j_2})]\|) \\ &\leqslant \sum_{j_1=1, j_2=1}^{k} E_{\mathrm{f}_i^*}^{1/2}(\|P'_{j_1}(X - m_i^*)^{j_1} - E_{\mathrm{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})\|^2) \\ &\quad \times E_{\mathrm{f}_i^*}^{1/2}\|P'_{j_2}(X - m_i^*)^{j_2} - E_{\mathrm{f}_i^*}(P'_{j_2}(X - m_i^*)^{j_2} | \mathrm{f}_i^*)\|^2). \end{aligned} \tag{40}$$

P ,

$$\begin{aligned} &E_{\mathrm{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1} - E_{\mathrm{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})\|^2) \\ &\quad = E_{\mathrm{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1}\|^2) - \|E_{\mathrm{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1})\|^2 \\ &\quad \leqslant E_{\mathrm{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1}\|^2) \leqslant \sqrt{n} E_{\mathrm{f}_i^*}(\|P'_{j_1}\|^2 \|X - m_i^*\|^{2j_1}) \\ &\quad = \sqrt{n}\|P'_{j_1}\|^2 E_{\mathrm{f}_i^*}(\|X - m_i^*\|^{2j_1}). \end{aligned} \tag{41}$$

B $P_i(x|\,\mathrm{f}_i^*) \leqslant U_i(x|\,\mathrm{f}_i^*)$,

$$E_{\mathrm{f}_i^*}(\|X - m_i^*\|^{2j_1}) \leqslant \int \|x - m_i^*\|^{2j_1} U_i(x|\,\mathrm{f}_i^*) \quad x$$

$$= \int \|y\|^{2j_1} w(y + m_i^*)(1^i \quad)^{-c_1} \quad ^{-\mathrm{r}(1/(1^i \quad)^{\mathrm{n}c_2})\|y\|^{c_2}} \quad y, \tag{42}$$

$y = x - m_i^*$. $w(x)$,

$$w(y + m_i^*) \leqslant w_0 + w_1\|y\| + \cdots + w_{k'}\|y\|^{k'}, \tag{43}$$

k' , $w_0, w_1, \ldots, w_{k'}$
$\|m_i^*\|$, . .,

$$w_i = w_0^i + w_1^i\|m_i^*\| + \cdots + w_{c_i}^i\|m_i^*\|^{c_i} \qquad i = 0, 1, \ldots, k', \tag{44}$$

$w_0^i, w_1^i, \ldots, w_{c_i}^i$, $c_0, \ldots, c_{k'}$
. B L 1,

$$w_i \leqslant v_0^i + v_1^i\|m_i^* - m_j^*\| + \cdots + v_{c_i}^i\|m_i^* - m_j^*\|^{c_i} \qquad i = 0, 1, \ldots, k', \tag{45}$$

$v_0^i, v_1^i, \ldots, v_{c_i}^i$. $w(y + m_i^*)$
E . (42),

$$E_{\mathrm{f}_i^*}(\|X - m_i^*\|^{2j_1}) \leqslant \sum_{l=0}^{k'} w_l(1^i \quad)^{-c_1} \int \|y\|^{2j_1+l} \quad ^{-\mathrm{r}(1/(1^i \quad)^{\mathrm{n}c_2})\|y\|^{c_2}} \quad y$$

$$= \sum_{l=0}^{k'} w_l(1^i \quad)^{-c_1+\mathrm{n}(2j_1+l+1)} \int \|u\|^{2j_1+l} \quad ^{-\mathrm{r}\|u\|^{c_2}} \quad u,$$

$u = y/(1^i \quad)^{\mathrm{n}}$. C , $\int \|u\|^{2j_1+l} \quad ^{-\mathrm{r}\|u\|^{c_2}} \quad u$
j_1 . 1^i ,
$E_{\mathrm{f}_i^*}(\|X - m_i^*\|^{2j_1})$ $\|m_i^* - m_j^*\|$.
M , P'_{j_1} $m_{i1}^*, \ldots, m_{in}^*$, $\|P'_{j_1}\|$
$\|m_i^*\|$. , $E_{\mathrm{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1}\|^2)$
$\|m_i^* - m_j^*\|$.
A , $E_{\mathrm{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1} - E_{\mathrm{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})\|^2)$
$\|m_i^* - m_j^*\|$. $\|m_i^* - m_j^*\| \geqslant T'$,

$$E_{\mathrm{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1} - E_{\mathrm{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})\|^2) \leqslant \mathrm{c}_{j_1}\|m_i^* - m_j^*\|^{p_{j_1}}, \tag{46}$$

c_{j_1} p_{j_1} .
E . (46) E . (40),

$$E_{\mathrm{f}_i^*}(\|t_i(X) - \mathrm{f}_i^*\|^2) \leqslant \mathrm{c}\,\|m_i^* - m_j^*\|^p, \tag{47}$$

c p . , E . (37), ( )
.

A ( ), $j \neq i$, $f'_j = E_{f^*_j}(t_i(X))$

$$
\begin{aligned}
E_{f^*_j}(\|t_i(X) - f^*_i\|^2) &\leqslant E_{f^*_j}((\|t_i(X) - f'_j\| + \|f'_j - f^*_i\|)^2) \\
&= E_{f^*_j}(\|t_i(X) - f'_j\|^2 + 2\|t_i(X) - f'_j\|\|f'_j - f^*_i\| + \|f'_j - f^*_i\|^2) \\
&\leqslant E_{f^*_j}(2\|t_i(X) - f'_j\|^2 + 2\|f'_j - f^*_i\|^2) \\
&= 2E_{f^*_j}(\|t_i(X) - f'_j\|^2) + 2\|f^*_i - f'_j\|^2. \qquad (48)
\end{aligned}
$$

I ,

$$E_{f^*_j}(\|t_i(X) - f'_j\|^2) \leqslant c_1\|m^*_i - m^*_j\|^{p_1}, \qquad (49)$$

c_1 p_1 . M ,

$$\|f^*_i - f'_j\| \leqslant \|f^*_i\| + \|f'_j\|.$$

B E . (38), $\|f^*_i\|$ $\|f'_j\|$ $c_2\|m^*_i - m^*_j\|^{p_2}$, c_2 p_2 . , E . (48), $E_{f^*_j}(\|t_i(X) - f^*_i\|^2)$ $\|m^*_i - m^*_j\|$. $\|m^*_i - m^*_j\| \geqslant T'$,

$$E_{f^*_j}(\|t_i(X) - f^*_i\|^2) \leqslant c_j\|m^*_i - m^*_j\|^{p_j}, \quad j \neq i, \qquad (50)$$

c_j p_j .

B E . (47) (50),

$$E(\|t_i(X) - f^*_i\|^2) = \sum_{j=1}^{K} a^*_j E_{f^*_j}(\|t_i(X) - f^*_i\|^2) \leqslant u M^q_i(F^*),$$

$M_i(F^*) =$ $_{j \neq i}\|m^*_i - m^*_j\|$, u q . □

Proof of Lemma 3.

$f(Z) = o(Z^p)$,

$Z \to 0$, p .

K

F^* $Z(F^*) = Z$. $i \neq j$, Z, m^*_{ij} m^*_i m^*_j

$$a^*_i P_i(m^*_{ij}|f^*_i) = a^*_j P_j(m^*_{ij}|f^*_j).$$

$$E_i = \{x : a^*_i P_i(x|f^*_i) \geqslant a^*_j P_j(x|f^*_j)\},$$

$$E_j = \{x : a^*_j P_j(x|f^*_j) > a^*_i P_i(x|f^*_i)\}.$$

A $Z(F^*)$, $(1^i\ )^n/(\|m^*_i - m^*_j\|)$ $(1^j\ )^n/(\|m^*_i - m^*_j\|)$. M , $k(S^*_i)$ $k(S^*_j)$. , (. .,) m^*_i (m^*_j) E_i (E_j). F , $\mathcal{N}_{r_i}(m^*_i)$ $\mathcal{N}_{r_j}(m^*_j)$ E_i E_j, , r_i r_j . $k(S^*_i)$ $k(S^*_j)$

, r_i r_j $\|m_i^* - m_j^*\|$ $\|m_i^* - m_j^*\|$. b_1
b_2

$$r_i \geqslant b_i \|m_i^* - m_j^*\| \qquad r_j \geqslant b_j \|m_i^* - m_j^*\|.$$

$$\mathscr{D}_i = \mathscr{N}^c_{r_i}(m_i^*) = \{x : \|x - m_i^*\| \geqslant r_i\},$$
$$\mathscr{D}_j = \mathscr{N}^c_{r_j}(m_j^*) = \{x : \|x - m_j^*\| \geqslant r_j\}$$

$$E_i \subset D_j, \qquad E_j \subset D_i.$$

M , $e_{ij}(\mathrm{F}^*)$ $h_k(x)$

$$\begin{aligned} e_{ij}(\mathrm{F}^*) &= \int h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} \\ &= \int_{E_i} h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} + \int_{E_j} h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} \\ &\leqslant \int_{\mathscr{D}_j} h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} + \int_{\mathscr{D}_i} h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} \\ &\leqslant \int_{\mathscr{D}_j} h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} + \int_{\mathscr{D}_i} h_i(x)P(x|\mathrm{F}^*)\ \mathrm{m} \\ &= \mathrm{a}_j^* \int_{\mathscr{D}_j} P_j(x|\,\mathrm{f}_j^*)\ \mathrm{m} + \mathrm{a}_i^* \int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m}. \end{aligned}$$

$\int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m}.$ $r_i \geqslant b_i \|m_i^* - m_j^*\|,$

$$\int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m} \leqslant \int_{\|x - m_i^*\| \leqslant b_i \|m_i^* - m_j^*\|} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m}.$$

B $y = (x - m_i^*)/\|m_i^* - m_j^*\|,$

$$\begin{aligned} &\int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m} \\ &\quad \leqslant \int_{\|y\| \leqslant b_i} w(\|m_i^* - m_j^*\|y + m_i^*)(1^i \quad)^{-c_1} \ {}^{-\mathrm{r}(\|m_i^* - m_j^*\|^{c_2})/(1 \quad)^{\mathrm{n}c_2}\|y\|^{c_2}} \|m_i^* - m_j^*\|\ \ \mathrm{m}' \\ &\quad = \int_{\|y\| \leqslant b_i} \|m_i^* - m_j^*\| w(\|m_i^* - m_j^*\|y + m_i^*)(1^i \quad)^{-c_1} \\ &\qquad \times \ {}^{-\mathrm{r}(\|m_i^* - m_j^*\|^{c_2})/(1 \quad)^{\mathrm{n}c_2}\|y\|^{c_2}}\ \ \mathrm{m}', \end{aligned} \tag{51}$$

m' m .
$w(\|m_i^* - m_j^*\|y + m_i^*)$
m_i^* $\|m_i^* - m_j^*\|,$
q
$\|m_i^* - m_j^*\|^{-q} w(\|m_i^* - m_j^*\|y + m_i^*)$ $\mathrm{Z}(\mathrm{F}^*) \to 0.$

$\|m_i^* - m_j^*\|^{-q} w(\|m_i^* - m_j^*\| y + m_i^*)$
y . M , L 1,

$$\|m_i^* - m_j^*\|^{1+q}(1^i \quad)^{-c_1} \leqslant O(\mathrm{Z}^{-c_1'}),$$
$$\|m_i^* - m_j^*\|^{c_2}(1^i \quad)^{-\mathrm{n}c_2} \geqslant O(\mathrm{Z}^{-c_2}),$$

$c_1' = (q+1) \vee (c_1/\mathrm{n})$.
A , E . (51)

$$\int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*) \quad \mathrm{m} \leqslant \int_{\mathscr{B}_i} \frac{1}{\mathrm{Z}^{c_1'}(\mathrm{F}^*)} w_1(y) \quad {}^{-\mathrm{r}'(1/\mathrm{Z}^{c_2}(\mathrm{F}^*))\|y\|^{c_2}} \quad \mathrm{m}'$$
$$= \int_{\mathscr{B}_i} \frac{1}{\mathrm{Z}^{c_1'}} w_1(y) \quad {}^{-\mathrm{r}'(1/\mathrm{Z}^{c_2})\|y\|^{c_2}} \quad \mathrm{m}', \tag{52}$$

$\mathscr{B}_i = \{y : \|y\| \geqslant b_i\}$, r' , $w_1(y)$
y .
F ,

$$F_i(\mathrm{Z}) = \int_{\mathscr{B}_i} P(y|\mathrm{Z}) \quad y, \quad P(y|\mathrm{Z}) = \frac{1}{\mathrm{Z}^{c_1'}} w_1(y) \quad {}^{-\mathrm{r}'(1/\mathrm{Z}^{c_2})\|y\|^{c_2}}$$

$F_i(\mathrm{Z})/\mathrm{Z}^p$ Z .
F $y \in \mathscr{B}_i$,

$$\begin{aligned}
{}_{\mathrm{Z}\to 0} \frac{P(y|\mathrm{Z})}{\mathrm{Z}^p} &= w_1(y) {}_{\mathrm{Z}\to 0} \frac{1}{\mathrm{Z}^{(c_1'+p)}} \quad {}^{-\mathrm{r}'(1/\mathrm{Z}^{c_2})\|y\|^{c_2}} \\
&= w_1(y) {}_{\mathrm{z}=\frac{1}{\mathrm{Z}}\to\infty} \frac{\mathrm{z}^{(c_1'+p)}}{\mathrm{z}^{c_2}\,\mathrm{r}'\|y\|^{c_2}} \\
&= 0,
\end{aligned}$$

$\mathscr{B}_i$,

$$\begin{aligned}
{}_{\mathrm{Z}\to 0} \frac{F_i(\mathrm{Z})}{\mathrm{Z}^p} &= {}_{\mathrm{Z}\to 0} \int_{\mathscr{B}_i} \frac{P(y|\mathrm{Z})}{\mathrm{Z}^p} \quad \mathrm{m}' \\
&= \int_{\mathscr{B}_i} {}_{\mathrm{Z}\to 0} \frac{P(y|\mathrm{Z})}{\mathrm{Z}^p} \quad \mathrm{m}' \\
&= 0
\end{aligned}$$

$F_i(\mathrm{Z}) = o(\mathrm{Z}^p)$. I E . (52)

$$_{\mathrm{Z}(\mathrm{F}^*)=\mathrm{Z}} \int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*) \quad \mathrm{m} = o(\mathrm{Z}^p). \tag{53}$$

, :

$$_{\mathrm{Z}(\mathrm{F}^*)=\mathrm{Z}} \int_{\mathscr{D}_j} P_j(x|\,\mathrm{f}_j^*) \quad \mathrm{m} = o(\mathrm{Z}^p).$$

A ,

$$f_{ij}(\mathrm{Z}) = \sum_{\mathrm{Z}(\mathrm{F}^*)=\mathrm{Z}} e_{ij}(\mathrm{F}^*)$$

$$\leqslant \sum_{\mathrm{Z}(\mathrm{F}^*)=\mathrm{Z}} \left(\mathrm{a}_j^* \int_{\mathscr{D}_j} P_j(x|\,\mathrm{f}_j^*) \quad \mathrm{m} + \mathrm{a}_i^* \int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*) \quad \mathrm{m} \right)$$

$$\leqslant \sum_{\mathrm{Z}(\mathrm{F}^*)=\mathrm{Z}} \int_{\mathscr{D}_j} P_j(x|\,\mathrm{f}_j^*) \quad x + \sum_{\mathrm{Z}(\mathrm{F}^*)=\mathrm{Z}} \int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*) \quad \mathrm{m})$$

$$= o(\mathrm{Z}^p).$$

,

$$f(\mathrm{Z}) \leqslant \sum_{ij} f_{ij}(\mathrm{Z}) = o(\mathrm{Z}^p). \tag{54}$$

M ,

$$\lim_{\mathrm{Z}\to 0} \frac{f^{\mathrm{e}}(\mathrm{Z})}{\mathrm{Z}^p} = \lim_{\mathrm{Z}\to 0} \left(\frac{f(\mathrm{Z})}{\mathrm{Z}^{\frac{p}{\mathrm{e}}}} \right)^{\mathrm{e}} = 0,$$

$f^{\mathrm{e}}(\mathrm{Z}) = o(\mathrm{Z}^p)$ $f^{\mathrm{e}}(\mathrm{Z}(\mathrm{F}^*)) = o(\mathrm{Z}^p(\mathrm{F}^*))$. □

References

1 O.E. B -N , I E F , , N , 1978.
2 B. D , M. L , E. M , C EM , A . . 27 (1999) 94 128.
3 A.P. D , N.M. L , D.B. , M EM , J. . . B 39 (1977) 1 38.
4 .C. H , E EM , : P C , A A , 1987, . 266 271.
5 M. J , .I. J , C EM , J. A . . A . 88 (1993) 221 228.
6 M. J , .I. J , A EM -N , J. . . B 59 (1997) 569 587.
7 M.I. J , .A. J , H EM , N C . 6 (1994) 181 214.
8 M.I. J , L. , C EP , N N 8 (1995) 1409 1431.
9 N. L , N. L , D. , M : EM , J. A . . A . 82 (1987) 97 105.
10 K. L , A -N EM , . . 5 (1995) 1 18.
11 E.L. L , P E , , N , 1985.
12 C. L , D.B. , ECME : A EM ECM

16 .L. M , D. D , EM _ - , J. . .
. B 59 (1997) 511 567.
17 L. . , A H M ,
P . IEEE 77 (1989) 257 286.
18 .A. , H.F. , M , , EM , IAM .
26 (1984) 195 239.
19 C.F.J. , O EM , A . . 11 (1983) 95 103.
20 L. , M.I. J , O EM G , N
C . 8 (1996) 129 151.

Jinwen Ma M
' J 1988 P .D.
N 1992. F J 1992 N 1999,
L A D M ,
. F D 1999,
I M , . I 2001,
D I
M , P . D 1995 2004,
D C & E ,
C H K A F . H
50 , , ,
.

Lei Xu (IEEE F) C & E ,
C H K . H P .D.
1986, D M , P
1987
1988. D 1989 93,
F , C A, H MI . H C HK
1993 , 1996
2002. P .
, N N , IEEE N N ,
I N N (01-03),
C F C IEEE C I (01-03),
A -P N N A . P . 100
, , , ,
. H / / /
,
. H C
(1994 C N N A)
(1995 INN L A). P . IEEE F F I
A P , E A .