

Parallel Adaptive Algorithms for the Solution of the Algebraic Riccati Equation

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Abstract.

Parallel algorithms for the solution of the Algebraic Riccati Equation (ARE) are presented. The algorithms are based on the Sherman-Morrison-Woodbury formula and the parallelization of the matrix-vector multiplication. The algorithms are implemented on a multi-processor system. The numerical results show that the proposed algorithms are efficient and robust. The algorithms are implemented on a multi-processor system. The numerical results show that the proposed algorithms are efficient and robust.

1. Introduction

The Algebraic Riccati Equation (ARE) is a fundamental problem in control theory. It is given by

$$A^T X + X A - X B R^{-1} B^T X + Q = 0$$

where A, B, Q are $n \times n$, $n \times m$, and $n \times n$ matrices, respectively, and R is an $m \times m$ positive definite matrix. The solution X is a symmetric $n \times n$ matrix. The ARE is a nonlinear equation and its solution is not trivial. There are many algorithms for the solution of the ARE, but they are often computationally expensive. In this paper, we propose parallel algorithms for the solution of the ARE. The algorithms are based on the Sherman-Morrison-Woodbury formula and the parallelization of the matrix-vector multiplication. The algorithms are implemented on a multi-processor system. The numerical results show that the proposed algorithms are efficient and robust.

$$y = Wx + WAs + WA s$$

where $y \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, $W \in \mathbb{R}^{n \times m}$, $A \in \mathbb{R}^{n \times n}$, $s \in \mathbb{R}^m$, and WAs is a vector in \mathbb{R}^n . The proposed algorithms are based on the Sherman-Morrison-Woodbury formula and the parallelization of the matrix-vector multiplication. The algorithms are implemented on a multi-processor system. The numerical results show that the proposed algorithms are efficient and robust.

1. m, ..., s

$$J \mathbf{R} = \sum_{i=1}^n \sum_{j=1}^n r_{ij}^4 \nu_j^s k_i^m = \sum_{i=1}^n \sum_{j=1}^n r_{ij}^4 k_{ij}.$$

$$k_1^m \geq \dots \geq k_p^m > \dots > k_{p+1}^m \geq \dots \geq k_n^m \quad \nu_1^s \geq \dots \geq \nu_p^s > \nu_{p+1}^s \geq \dots \geq \nu_n^s$$

$J \mathbf{R}$ is a symmetric matrix. The eigenvalues of $J \mathbf{R}$ are $\lambda_1, \lambda_2, \dots, \lambda_n$. The corresponding eigenvectors are $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. The matrix $J \mathbf{R}$ is positive definite if and only if all the eigenvalues are positive. The matrix $J \mathbf{R}$ is negative definite if and only if all the eigenvalues are negative. The matrix $J \mathbf{R}$ is indefinite if it has both positive and negative eigenvalues. The matrix $J \mathbf{R}$ is singular if and only if at least one of the eigenvalues is zero.

Lemma 1. Suppose that $F(\mathbf{x})$ ($\mathbf{x} \in \mathbb{R}^m$) is a twice differentiable scalar function under the following constraints:

$$C_i(\mathbf{x}) = 0, \quad i = 1, \dots, k.$$

Construct a Lagrange function with a Lagrange multiplier set $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$, i.e., $L(\mathbf{x}, \lambda) = F(\mathbf{x}) - \sum_{i=1}^k \lambda_i C_i(\mathbf{x})$, and assume that \mathbf{x}^*, λ^* is a solution of the system of the equalities that all the derivatives of $L(\mathbf{x}, \lambda)$ with respect to the variables of \mathbf{x} and the Lagrange multipliers λ_i are equal to zeros. It is also assumed that these $\nabla C_i(\mathbf{x}^*)$ are linearly independent. If for any nonzero vector \mathbf{q} under the constraints $\mathbf{q}^T \nabla_{\mathbf{x}} C_i(\mathbf{x}^*) = 0$ for $i = 1, \dots, k$, we have

$$\mathbf{q}^T \nabla_{\mathbf{x}}^2 L(\mathbf{x}^*, \lambda^*) \mathbf{q} < 0 \quad \text{or} > 0,$$

then \mathbf{x}^* is a local maximum (or local minimum) of $F(\mathbf{x})$ under the constraints.

$\mathbf{q}^T \nabla_{\mathbf{x}}^2 L(\mathbf{x}^*, \lambda^*) \mathbf{q} < 0$

2.

$J \mathbf{R}$ is a symmetric matrix. The eigenvalues of $J \mathbf{R}$ are $\lambda_1, \lambda_2, \dots, \lambda_n$. The corresponding eigenvectors are $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. The matrix $J \mathbf{R}$ is positive definite if and only if all the eigenvalues are positive. The matrix $J \mathbf{R}$ is negative definite if and only if all the eigenvalues are negative. The matrix $J \mathbf{R}$ is indefinite if it has both positive and negative eigenvalues. The matrix $J \mathbf{R}$ is singular if and only if at least one of the eigenvalues is zero.

\mathbf{I} , \dots A

$\mathbf{R}\mathbf{R}^T$

$$L(\mathbf{R}, \lambda) = \sum_{i=1}^n \sum_{j=1}^n r_{ij}^4 k_{ij} + \sum_{i=1}^n \sum_{j=i}^n \lambda_{ij} \left(\sum_{l=1}^n r_{li} r_{lj} - \delta_{ij} \right),$$

δ_{ij}

$$\frac{\partial L(\mathbf{R}, \lambda)}{\partial r} = 4k r^3 + \sum_{i=1}^1 r \lambda + 2\lambda r + \sum_{i=+1}^i r \lambda ; \tag{10}$$

$$\frac{\partial L(\mathbf{R}, \lambda)}{\partial \lambda} = \sum_{i=1}^i r r - \delta . \tag{11}$$

λ \dots \mathbf{U} $u_{ij} \ n \times n \cdot S$

$$u = \begin{cases} \lambda \end{cases}$$

Theorem 1. *If \mathbf{R}^* is a permutation matrix up to sign indeterminacy and $k_{ij} >$ at all the positions where $|r_{ij}^*|$, it corresponds to a local maximum of the objective function $J \mathbf{R}$.*

Proof: ... \mathbf{R} $n \times n$ \mathbf{m} $\times \mathbf{R}$ $n^2 \times$...

$$\text{vec } \mathbf{R} = [r_{11}, r_{21}, \dots, r_{n1}, r_{12}, r_{22}, \dots, r_{n2}, \dots, r_{1n}, r_{2n}, \dots, r_{nn}]^T \in \mathbb{R}^{n^2}.$$

... \mathbf{S} ... \mathbf{m} ... \mathbf{S} ... \mathbf{S} ... $n^2 \times$... \mathbf{q}

$$\mathbf{q} = [q_{11}, q_{21}, \dots, q_{n1}, q_{12}, q_{22}, \dots, q_{n2}, \dots, q_{1n}, q_{2n}, \dots, q_{nn}]^T \in \mathbb{R}^{n^2}.$$

... \mathbf{A} ... \mathbf{S}

$$\frac{\partial^2 L(\mathbf{R}, \boldsymbol{\lambda})}{\partial r_{ij} \partial r_{i'j'}} = \delta_{(i,j),(i',j')} k_{ij} r_{ij}^2 u_{jj},$$

$$\delta_{(i,j),(i',j')} = \begin{cases} 1 & \text{if } i=i' \text{ and } j=j' \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{U} = \mathbf{R}^T \mathbf{B}.$$

\mathbf{R} \mathbf{R}^* \mathbf{S} ... \mathbf{m} ... \mathbf{X} ... \mathbf{m} ... \mathbf{U}^* ... \mathbf{S} ...
 $\boldsymbol{\lambda}^*$... \mathbf{m} ... \mathbf{X} ... \mathbf{A} ... \mathbf{A} $k_{ij} >$ every
 $|r_{ij}^*|$... \mathbf{S} ... \mathbf{A} u_{jj}^*

$J \mathbf{R}$

Theorem 2. *If \mathbf{R}^* is a permutation matrix up to sign indeterminacy and $k_{ij} < |r_{ij}^*|$ at all the positions where $|r_{ij}^*|$, it corresponds to a local minimum of the objective function $J \mathbf{R}$.*

Remark 2.

\mathbf{R}^*

$J \mathbf{R}$

1. g ., # # , # . : - s g s #
2. . : # s s. s. g ss g, 41(7)(1993) 2461-2470
3. # ss . # . : A s s # s: A
4. A. # s . : A - 1 s -
5. A . , A., # g . : A g g s -
6. # s. A s # s # s g s. # (.),
7. s . : # s # s . g
8. # , # g . , # g ., A . : # s
9. # , # g . , A . : # s A
10. # , # g . , A . : # s s # s -
11. # s s g s. # # , 10(1998) 2103-2114
12. s . s . : # s: A #
13. g . # s. # # , 13(2001) 677-689
14. , - , g . : A s g
15. # g . . # . : g g # s
16. # . # , # . , # . : - - g # #
17. # . # , # . : A # s # A - - g # #
18. # # . . : A g g. - -
19. # A., A s . A., . : g # g s -