Contrast Functions for Non-circular and Circular Sources Separation in Complex-Valued ICA

Abstract— In this paper, the complex-valued ICA problem is studied in the context of blind complex-source separation. We formulate the complex ICA problem in a general setting, and define the superadditive functional that may be used for constructing a contrast function for circular complex sources separation. We propose several contrast functions and study their properties. Finally, we also discuss relevant issues and present the convex analysis of a specific contrast function.

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$$\mathbb{E}[\ ] = \mathbb{E}[\quad + \quad] = \mathbb{E}[\quad] + \ \mathbb{E}[\quad].$$

- :

$$\mathbb{E}[\ ^2] = \mathbb{E}[\ ^2\ ] - \mathbb{E}[\ ^2\ ] + 2\ \mathbb{E}[\qquad].$$

- ( ):

$$\mathtt{var}[\ ] = \mathbb{E}[|\ \ - \mathbb{E}[\ ]|^2] = \mathbb{E}[|\ |^2] - \big|\mathbb{E}[\ ]\big|^2.$$

$C_{ij} = \mathbb{E}\big[(\ _i - \mathbb{E}[\ _i])(\ ^*_j - \mathbb{E}[\ ^*_j])\big] = \mathbb{E}[\ _i\ ^*_j] - \mathbb{E}[\ _i]\mathbb{E}[\ ^*_j]$. ($\neq i$) *uncorrelated* $C_{ij} = 0$.

$\mathbf{z} = [\ _1, \ldots, \ _n]$, $\mathbf{z}^H = [\ ^*_1, \ldots, \ ^*_n]^T \equiv (\mathbf{z}^*)^T$ (. .,), $\mathbb{E}[\mathbf{z}]$, $\mathtt{cov}[\mathbf{z}] \equiv \mathbb{E}\big[(\mathbf{z} - \mathbb{E}[\mathbf{z}])(\mathbf{z} - \mathbb{E}[\mathbf{z}])^H\big]$; , *pseudo-covariance matrix* $\mathtt{pcov}[\mathbf{z}] \equiv \mathbb{E}\big[(\mathbf{z} - \mathbb{E}[\mathbf{z}])(\mathbf{z} - \mathbb{E}[\mathbf{z}])^T\big]$.

Definition 1: A - α, ( ) ($e^{j\alpha}$) (. ., ( )). $\mathbf{z}$ *second-order circular* - (. .,); $\mathbb{E}[\ ] = 0$, - $\mathbb{E}[\ ^2] = 0$ . . $\mathbb{E}[\mathbf{z}\mathbf{z}^H]$, $\mathbf{z}$; $\mathbf{z}$ *strongly uncorrelated* $\mathbb{E}[\mathbf{z}\mathbf{z}^H]$ $\mathbb{E}[\mathbf{z}\mathbf{z}^T]$. $\mathbf{z}$, $\mathbf{z}$ *symmetric*; $\mathbf{z}$, $\mathbf{z}$ - , $\mathbb{E}[\mathbf{z}\mathbf{z}^T] = \mathbf{0}$.

, *zero-mean* 1 :

$$\mathtt{skewness}(\ ) = \mathbb{E}[|\ |^3]/\big(\mathbb{E}[|\ |^2]\big)^{3/2}, \tag{1}$$

$$\mathtt{kurtosis}(\ ) = \mathbb{E}[|\ |^4] - 2\big(\mathbb{E}[|\ |^2]\big)^2 - \big|\mathbb{E}[\ ^2]\big|^2. \tag{2}$$

Definition 2: $(\ _1, \ _2) = \ ^1(\ _1)\ ^2(\ _2)$; 1 2 , $(|\ _1|, |\ _2|) = (|\ _1|)\ (|\ _2|)$.

$J(\mathbf{z})$ - , - $\mathbf{z} \in \mathbb{C}^N$, :

$$\nabla J \equiv \frac{\partial J(\mathbf{z})}{\partial \mathbf{z}^*} = \frac{1}{2}\Big(\frac{\partial J(\mathbf{z})}{\partial \mathbf{z}} + \ \frac{\partial J(\mathbf{z})}{\partial \mathbf{z}}\Big) = 0.$$

, $\frac{\partial J(\mathbf{z})}{\partial \mathbf{x}} = \frac{\partial J(\mathbf{z})}{\partial \mathbf{z}} = 0$.

Definition 3: A - $J(\mathbf{z})$ ($\mathbf{z} \in \mathbb{C}^N$)

$$J\big(\lambda \mathbf{z}_1 + (1-\lambda)\mathbf{z}_2\big) \leq \lambda J(\mathbf{z}_1) + (1-\lambda)J(\mathbf{z}_2)$$

$\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{C}^N$ $0 \leq \lambda \leq 1$. A , , $\mathbf{H} = \frac{\partial^2 J(\mathbf{z})}{\partial \mathbf{z} \partial \mathbf{z}^H}$, (. .,), $J(\mathbf{z})$.

A A, - A : $\mathbf{x} = \mathbf{A}\mathbf{s}$, $\mathbf{s} \in \mathbb{C}^n$ - - , - , $\mathbf{x} \in \mathbb{C}^n$ - - , $\mathbf{A} \in \mathbb{C}^{n \times n}$ (. .,) - .

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$\mathbb{E}[\mathbf{x}] = 0$ $\mathbb{E}[\mathbf{x}\mathbf{x}^H] = \mathbf{I}$,

strong-uncorrelating transform 10, 11 .

, *mutual information*, -

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\begin{aligned}
I(\ _1,\dots,\ _n) &= \mathbb{E}_{p(\mathbf{y})}\Big[\log \frac{(\mathbf{y})}{\prod_{i\ 1}^{n} (\ _i)}\Big] \\
&= \int (\mathbf{y}) \log \frac{(\mathbf{y})}{\prod_{i\ 1}^{n} (\ _i)} d\mathbf{y} \qquad (4)
\end{aligned}
$$

(4) $\{\ _1,\dots,\ _n\}$. ? $\{\ _1,\cdots,\ _n\}$, (4)

$$I(\ _1,\dots,\ _n) = -\frac{1}{2}\log\Big(\frac{(\mathbf{C_y})}{\prod_{i\ 1}^{n} C_{ii}}\Big) = -\frac{1}{2}\sum_{i\ 1}^{n}\log(\lambda_i) \quad (5)$$

$C_{ij} = \mathbb{E}[(\ _i - \mathbb{E}[\ _i])(\ _j^* - \mathbb{E}[\ _j^*])]$, λ_i 0 : $\mathbf{C_y u} = \lambda \mathbf{\Lambda u}$, $\mathbf{C_y} = \mathrm{cov}[\mathbf{y}]$, $\mathbf{\Lambda} =$ $\{C_{11}, C_{22}, \dots, C_{nn}\}$.

$$I(\ _1,\dots,\ _n) = \sum_{i\ 1}^{n} H(\ _i) - H(\mathbf{y}),$$

$H(\mathbf{y})$ - $\{\ _1,\dots,\ _n\}$; $H(\mathbf{y}) = H(\mathbf{x}) + \log|\ (\mathbf{W})|$, (4)

$$J(\mathbf{W}) = \sum_{i\ 1}^{n} H(\ _i) - \log|\ (\mathbf{W})| - H(\mathbf{x}), \quad (6)$$

$H(\mathbf{x})$ (. . .) $\mathbf{W}$. (6) . . . $\mathbf{W}^*$

$$\nabla_{\mathbf{W}^*} J(\mathbf{W}) = \Big(\mathbb{E}_{\mathbf{y}}[\psi(\mathbf{y})\mathbf{y}^H] - \mathbf{I}\Big)\mathbf{W}^{-H}, \quad (7)$$

$\mathbf{W}^{-H}$ $\mathbf{W}^{-1}$, $\psi(\mathbf{y}) = [\psi(\ _1),\dots,\psi(\ _n)]^T$:

$$
\begin{aligned}
\psi(\ _i) &= -\frac{d\log\ (\ _i)}{d\ _i^*} = -\frac{\frac{\partial p(y_i)}{\partial y_i} + \frac{\partial p(y_i)}{\partial y_i}}{(\ _i)} \\
&= \frac{\partial \log\ (\ _i)}{\partial\ _i} + \frac{\partial \log\ (\ _i)}{\partial\ _i}, \qquad (8)
\end{aligned}
$$

$\psi(\cdot)$ *complex score function.* , $(\ _i) \equiv (\ _i,\ _i)$ (8) $_i$ $_i$.

(6), (2) - 7, 13 :

$$\Delta\mathbf{W} = \eta\big(\mathbf{I} - \psi(\mathbf{y})\mathbf{y}^H\big)\mathbf{W}. \quad (9)$$

Liouville's theorem, *boundedness* *analyticity* ; $\psi(\cdot)$ (. ., 7 - 9 , 13 , 14); , .

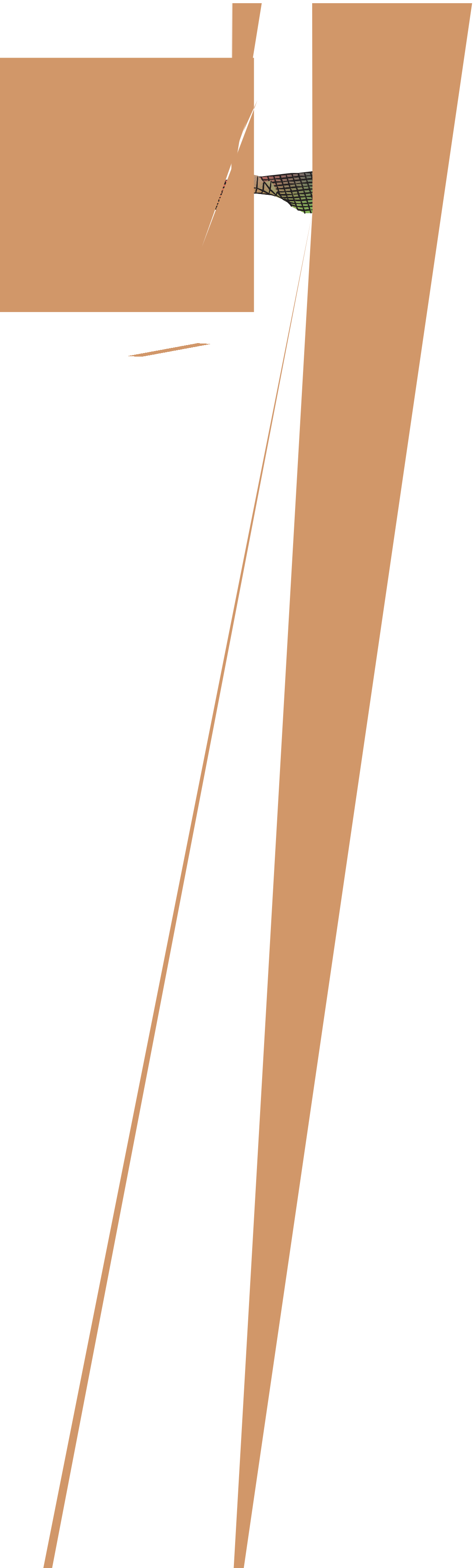
, - . , $e_i = \ _i - \ _i$, $H(\ _i \mid \ _i)$

$$H(\ _i \mid \ _i) = H(\ _i - \ _i) \equiv H(e_i) \quad (10)$$

, $H(\ _i) \equiv H(\ _i,\ _i)$ -

$$
\begin{aligned}
H(\ _i) &= H(\ _i \mid \ _i) + H(\ _i) \equiv H(-e_i) + H(\ _i) \\
&= H(\ _i \mid \ _i) + H(\ _i) \equiv H(e_i) + H(\ _i) \qquad (11)
\end{aligned}
$$

? e_i - ; $_i$ $_i$



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$ij \neq 0$.

$Q(\ i) > 0$ **R** ,

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Corollary 1: Q

$$Q(\ _1 + \ _2) = Q(\ _1) + Q(\ _2) \tag{19}$$

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$_1$ $_2$; (17)

$Q(\ _j) > 0$

$_j$.

Proof: ? (19)

. Q , (19)

$$\begin{aligned} Q^2(\ _1 + \ _2) &= Q^2(\ _1) + Q^2(\ _2) + 2Q(\ _1)Q(\ _2) \\ &\geq Q^2(\ _1) + Q^2(\ _2), \end{aligned}$$

$Q(\ _1) = Q(\ _2) = 0.$

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2 1 *sufficient*

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not $\mathrm{cum}_k(\ _1 + \ _2) = \mathrm{cum}_k(\ _1) + \mathrm{cum}_k(\ _2)$

$_1$ $_2$.

. , $_1 = |\ _1|e^{j\theta_1}$, $_2 = |\ _2|e^{j\theta_2}$

$(\theta_1, \theta_2 \in \mathbb{R})$,

$$\begin{aligned} _1 + \ _2 &= |\ _1|e^{j\theta_1} + |\ _2|e^{j\theta_2} \\ &= \sqrt{|\ _1|^2 + |\ _2|^2 + 2|\ _1|\,|\ _2|\cos(\theta_1 - \theta_2)}\,e^{j\theta}, \end{aligned}$$

$\theta = \arctan \frac{_1 \quad \theta_1 + \ _2 \quad \theta_2}{_1 \quad \theta_1 + \ _2 \quad \theta_2}$. , $\theta_1 - \theta_2$, $||\ _1| - |\ _2|| \leq |\ _1 + \ _2| \leq |\ _1| + |\ _2|$.

Theorem 3: Q (2-)

; $= |\ |e^{j\theta}$ $(\theta \in \mathbb{R})$,

$\overline{Q}(\ ) = Q(|\ |) = \sqrt{Q^2(|\ |\cos\theta) + Q^2(|\ |\sin\theta)}$

(2-)

.

Proof: . $_1 = |\ _1|e^{j\theta_1}$

$_2 = |\ _2|e^{j\theta_2}$

; $= |\ |e^{j\theta} = \ _1 + \ _2$, $=$

$_1 + \ _2$ $= \ _1 + \ _2$;

Q ,

$$\begin{aligned} Q^2(|\ |\cos\theta) &\geq Q^2(|\ _1|\cos\theta_1) + Q^2(|\ _2|\cos\theta_2) \\ Q^2(|\ |\sin\theta) &\geq Q^2(|\ _1|\sin\theta_1) + Q^2(|\ _2|\sin\theta_2). \end{aligned}$$

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$$\begin{aligned} &\overline{Q}^2(\ ) = Q^2(|\ |) = Q^2(|\ |\cos\theta) + Q^2(|\ |\sin\theta) \\ &\geq Q^2(|\ _1|) + Q^2(|\ _2|) = \overline{Q}^2(\ _1) + \overline{Q}^2(\ _2), \end{aligned} \tag{20}$$

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Lemma 1: $Q(\ )$ $(\ \in \mathbb{C})$ -

; $\tilde{\mathbf{z}} = [\ \ ,\ \ ]$, Q

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$\tilde{\mathbf{z}} \in \mathbb{R}^2$; , ( ) :

$Q(\boldsymbol{\alpha} + \tilde{\mathbf{z}}) = Q(\tilde{\mathbf{z}})$ $(\forall \boldsymbol{\alpha} \in \mathbb{R}^2)$; ( ) :

$Q(\alpha\tilde{\mathbf{z}}) = |\alpha|Q(\tilde{\mathbf{z}})$ $(\forall \alpha \in \mathbb{R})$.

Proof: .

. $\alpha =$

$|\alpha|e^{j0}$ (. .,). ? . □

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Theorem 4: , Q -

≥ 2,

$-\sum_{i\ 1}^{n} Q^k(\ _i)$ $-\sum_{i\ 1}^{n} Q^{2k}(\ _i)$

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> 2 $\{\ _j\}$,

$Q(\ _j) = 0$.

Proof: 15

. $= 2$ $\mathbf{R} = \mathbf{WA}$, -

Q, $Q^2(\ _i) \leq \sum_{j\ 1}^{n} |\ _{ij}|^2 Q^2(\ _j) \leq$

$\sum_{j\ 1}^{n} Q^2(\ _j)$ $|\ _{ij}| \leq \sum_{k\ 1}^{n} |\ _{ik}|^2 = 1$; $>$

2, $Q^k(\ _i) \leq \sum_{j\ 1}^{n} |\ _{ij}|^k Q^k(\ _j) \leq \sum_{j\ 1}^{n} Q^k(\ _j)$.

$-\sum_{i\ 1}^{n} Q^k(\ _i) \geq -\sum_{j\ 1}^{n} Q^k(\ _j)$,

$-\sum_{i\ 1}^{n} Q^k(\ _i)$.

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Remark:

$\mathbf{0}$: $\mathbb{E}[\mathbf{x}\mathbf{x}^H] =$

$\mathbf{A}\mathbb{E}[\mathbf{s}\mathbf{s}^H]\mathbf{A}^H = \mathbf{U}\boldsymbol{\Sigma}\mathbf{U}^H$, $\boldsymbol{\Sigma}$

- , $\mathbf{U}$

. $\mathbf{z} = \mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{x} =$

$\mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{A}\mathbf{s} \equiv \tilde{\mathbf{A}}\mathbf{s}$, $\tilde{\mathbf{A}} = \mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{A}$

$\mathbb{E}[\mathbf{z}\mathbf{z}^H] = \tilde{\mathbf{A}}\mathbb{E}[\mathbf{s}\mathbf{s}^H]\tilde{\mathbf{A}}^H = \mathbf{I}$;

$\tilde{\mathbf{A}}$ $\mathbb{E}[\mathbf{s}\mathbf{s}^H] = \mathbf{I}$.

B. Examples of Contrast Functions

1) Range Function: 15 21 ,

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, $R( )$,

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$$R( ) = d, \quad \{ ( ) \geq 0 \mid | - {}_0| \leq d\} \tag{21}$$

$_0$, $d \in \mathbb{R}^+$.

Theorem 5: -

$_1$ $_2$,

$$R({}_1 + {}_2) = R({}_1) + R({}_2). \tag{22}$$

Proof: $| {}_1 + {}_2| \leq | {}_1| + | {}_2|$. □

Corollary 2: $= \alpha {}_1 + \beta {}_2$, $_1$ $_2$

, $\alpha, \beta \in \mathbb{C}$,

$$R( ) = |\alpha| \cdot R({}_1) + |\beta| \cdot R({}_2). \tag{23}$$

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$\alpha {}_1$ ($\beta {}_2$) $|\alpha|$ ($|\beta|$) $_1$ ($_2$),

α (β).

$Q({}_i) = R({}_i)$, (17)

$$J(\mathbf{W}) = \sum_{i\ 1}^{n} \log \Big(\sum_{j\ 1}^{n} | {}_{ij}| R({}_j) \Big) - \log | \ (\mathbf{W})|. \tag{24}$$

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Definition 6:

, $\{ : | | \leq +\infty\}$;

$\int_0^d (| |) d| | \approx 1 \quad (0 < d < +\infty)$.

, d

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2) Shannon Entropy Function: A -

- , $H({}_i)$.

$$\overline{Q}({}_i) = Q(| {}_i|) = \exp(H(| {}_i|)), \tag{25}$$

entropy power inequality, (25)

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(25) (17) (6)

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$\{ {}_i\}$, $\{| {}_i|\}$

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3) Rényi Entropy Function: -

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$(0 < \ \in \mathbb{R})$:

$$H_k({}_i) = \frac{1}{1 - } \log \Big(\int ({}_i)^k d {}_i \Big). \tag{26}$$

$\rightarrow 1$,

. $= 2$,

2 *extension entropy*

$$H_2({}_i) = -\log \Big(\int ({}_i)^2 d {}_i \Big).$$

Jensen inequality, $H_2({}_i) \leq H({}_i)$.

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$H_k({}_i) \geq H_r({}_i)$ $>$.

Lemma 2: $\{ : ( ) = g(| |)\}$, ,

$\exp(H_2( ))$, .

Proof: , $H_2( ) = H_2(+ \alpha)$

α. ? ,

$$H_2(\alpha) = -\log \Big(\int g(|\alpha |)^2 d|\alpha | \Big)$$
$$= -\log \Big(\int g(| |)^2 d| | \Big) + \log|\alpha|.$$

$\exp(H_2(\alpha)) = |\alpha| \exp(H_2( ))$. ,

$\exp(H_2( ))$. □

, $\overline{Q}({}_i) = Q(| {}_i|) = \exp(H_2(| {}_i|)) = \frac{1}{\int p(|y_i|)^2 dy_i}$,

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26,27 ; $\overline{Q}$

. $\{| {}_i|\}$,

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4) Fisher Information Function:

$$\mathbf{G} = \mathbb{E}[\psi( )^2] = \mathbb{E}\Big[\Big(\frac{d \log ( )}{d}\Big)^2\Big], \tag{27}$$

$\psi( ) = \frac{d \quad p( )}{d} = \frac{p'( )}{p( )}$.

$Q( ) = \mathbf{G}^{-1/2}$, Q

15 . ,

$$\overline{Q}( ) = \mathbb{E}\Big[\Big(\frac{\ '(| |)}{(| |)}\Big)^2\Big]^{-1/2}.$$

- , -

$\psi_k( ) =$

$$Q(\ ) = \quad (\Sigma\ )^{1/2} \equiv \quad \Big(\mathrm{cov}[\tilde{\mathbf{z}}]\Big)^{1/2},$$

$$\begin{aligned}
Q(\omega\ ) &= \quad \Big(\mathrm{cov}[\tilde{\mathbf{z}}']\Big)^{1/2} = \quad \Big(\mathrm{cov}[\Omega\tilde{\mathbf{z}}]\Big)^{1/2} \\
&= \quad \Big(\Omega\mathrm{cov}[\tilde{\mathbf{z}}]\Big)^{1/2} = \quad (\Omega)^{1/2} \quad \Big(\mathrm{cov}[\tilde{\mathbf{z}}]\Big)^{1/2} \\
&= (\omega^2 \ + \omega^2\ )^{1/2} \quad (\Sigma\ )^{1/2} = |\omega| \quad (\Sigma\ )^{1/2}.
\end{aligned}$$

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$$(\ ) \approx \mathcal{N}(\ )\left(1 + \frac{\kappa^3}{\ }\mathcal{H}_3(\ ) + \frac{\kappa^4}{4}\mathcal{H}_4(\ )\right)$$