Contrast Functions for Non-circular and Circular Sources Separation in Complex-Valued ICA

*Abstract***— In this paper, the complex-valued ICA problem is studied in the context of blind complex-source separation. We formulate the complex ICA problem in a general setting, and define the superadditive functional that may be used for constructing a contrast function for circular complex sources separation. We propose several contrast functions and study their properties. Finally, we also discuss relevant issues and present the convex analysis of a specific contrast function.**

$$
\begin{array}{cccc}\n\bullet & \bullet & \bullet & \bullet & \bullet \\
\mathbb{E}[\] = \mathbb{E}[\] + \] = \mathbb{E}[\] + \mathbb{E}[\] \ .\n\end{array}
$$
\n
$$
\begin{array}{cccc}\n\mathbb{E}[\] = \mathbb{E}[\] - \mathbb{E}[\]^2] = \mathbb{E}[\]^2] = \mathbb{E}[\]^2] = \mathbb{E}[\]^2] = \mathbb{E}[\]^2. \\
\begin{array}{cccc}\n\mathbb{E}[\] = \mathbb{E}[\] - \mathbb{E}[\]^2] = \mathbb{E}[\]^2] = \mathbb{E}[\]^2] = \mathbb{E}[\]^2. \\
\mathbb{E}[\] = \mathbb{E}[\]^
$$

$$
\nabla J \equiv \frac{\partial J(\mathbf{z})}{\partial \mathbf{z}^*} = \frac{1}{2} \left(\frac{\partial J(\mathbf{z})}{\partial \mathbf{z}} + \frac{\partial J(\mathbf{z})}{\partial \mathbf{z}} \right) = 0.
$$

,
$$
\frac{\partial J(\mathbf{z})}{\partial \mathbf{x}} = \frac{\partial J(\mathbf{z})}{\partial \mathbf{z}} = 0.
$$

Definition 3: A **real-valued function** $J(\mathbf{z})$ **(z** ∈ \mathbb{C}^N) $J(\lambda \mathbf{z}_1 + (1 - \lambda)\mathbf{z}_2) \leq \lambda J(\mathbf{z}_1) + (1 - \lambda)J(\mathbf{z}_2)$ $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{C}^N \qquad 0 \leq \lambda \leq 1.$ A A , is $H = \frac{\partial^2 J(\mathbf{z})}{\partial \mathbf{z} \partial \mathbf{z}^H}$, semidefinite (i.e., with non-negative real eigenvalues), then \mathcal{E} $J(\mathbf{z})$ is a convex function. \cdot \cdot $\bar{\chi}$ $\bar{\psi}$ ^A A \blacksquare similar vein in real-valued in real-valued \blacksquare $s \in \mathbb{C}^n$ and \mathbf{A} if $\mathbf{x} = \mathbf{A}\mathbf{s}$, $\mathbf{s} \in \mathbb{C}^n$ element source vector, $\mathbf{x} \in \mathbb{C}^n$ the d -dimensional complex-valued vector of \mathcal{A} $\mathbf{A} \in \mathbb{C}^{n \times n}$ (i.e., with function $\mathbf{A} \in \mathbb{C}^{n \times n}$) complex-valued mixing matrix. It is noted that the three that there are the are there are the three t types of index-valued in complex-valued \mathbf{A} : \bullet \bullet \bullet \bullet ; \bullet • Sign and scaling index \mathbf{S} ; \bullet Phase index index index in The first two indeterminacies are shared with the real-first two index σ \mathbf{A} ; whereas the phase ambiguity arises from the phase a inherent nature of the complex-valued data. The second and second a , when combined together, is referred to $\mathcal{S}_{\mathcal{A}}$ complex scale ambiguity. \mathcal{I} the context of approaches: the context of approaches: the approaches (i) setcomes (define the flation of $($ $\left($ referred to as \mathbf{A} ; and (ii) since \mathbf{A} is an operator separator separat aration approach, which separates all independent sources all independent sou at the same time. In this paper, we will focus on the same time. In this paper, we will focus on the same time simultaneous separation approach; the one-unit complex $\mathbf A$ can be regarded as a special case of simultaneous separation 20 .

 $\mathcal{I}(\mathcal{I})$ terms of simultaneous separation, the goal of complex separation, the goal of complex separation, $\mathcal{I}(\mathcal{I})$

$$
\mathbb{E}[\mathbf{x}] = 0 \qquad \mathbb{E}[\mathbf{x}\mathbf{x}^H] = \mathbf{I},
$$

strong-uncorrelating transform 10, 11. $\mathcal{B}(\mathcal{A})$ specific contrast functions, let us first functions, let us first functions, $\mathcal{B}(\mathcal{A})$

formation,

$$
I(\ _1,\ldots,\ _n) = \mathbb{E}_{p(\mathbf{y})}\Big[\log \frac{(\mathbf{y})}{\prod_{i=1}^n\binom{\mathbf{y}}{i}}\Big] \\
= \int (\mathbf{y})\log \frac{(\mathbf{y})}{\prod_{i=1}^n\binom{\mathbf{y}}{i}}d\mathbf{y} \qquad (4)
$$
\n(4)

Leibler divergence criterion that measures the discrepancy between the joint probability and the product of marginal

(4)
\n
$$
\{1, \dots, n\}
$$

 (4)

$$
I(\ _1,\ldots,\ _n) = -\frac{1}{2}\log\left(\frac{(\mathbf{C}_\mathbf{y})}{\prod_{i=1}^n C_{ii}}\right) = -\frac{1}{2}\sum_{i=1}^n \log(\lambda_i) \quad (5)
$$

$$
C_{ij} = \mathbb{E}[(\ _i - \mathbb{E}[\ _i])(\ _j^* - \mathbb{E}[\ _j^*])], \quad \lambda_i
$$

$$
\mathbf{C}_\mathbf{y} = \text{cov}[\mathbf{y}], \quad \Lambda = \mathbb{V} \quad \{C_{11}, C_{22}, \ldots, C_{nn}\}.
$$

$$
I(\ _1,\ldots,\ _n)=\sum_{i=1}^nH(\ _i)-H(\mathbf{y}),
$$

$$
H(\mathbf{y})
$$

 $\{ 1, \ldots, n \}; \qquad H(\mathbf{y}) = H(\mathbf{x}) +$ log | (**W**)|, (4)

$$
J(\mathbf{W}) = \sum_{i=1}^{n} H(i) - \log |(\mathbf{W})| - H(\mathbf{x}), \quad (6)
$$

$$
H(\mathbf{x}) \qquad (\ldots) \mathbf{W}.\tag{6} \ldots
$$

 \mathbf{W}^*

$$
\nabla_{\mathbf{W}^*} J(\mathbf{W}) = \left(\mathbb{E}_{\mathbf{y}} [\psi(\mathbf{y}) \mathbf{y}^H] - \mathbf{I} \right) \mathbf{W}^{-H}, \tag{7}
$$

$$
\mathbf{W}^{-H} \qquad \qquad \mathbf{W}^{-1},
$$

$$
\psi(\mathbf{y}) = [\psi(\mathbf{1}), \dots, \psi(\mathbf{n})]^T
$$

:

$$
\psi(\iota) = -\frac{d \log(\iota_i)}{d \iota_i^*} = -\frac{\frac{\partial p(y_i)}{\partial y_i} + \frac{\partial p(y_i)}{\partial y_i}}{\iota_i}
$$
\n
$$
= \frac{\partial \log(\iota_i)}{\partial \iota_i} + \frac{\partial \log(\iota_i)}{\partial \iota_i}, \tag{8}
$$

 $\psi(\cdot)$ complex score function. $\psi(\cdot)$ = ($(i) \equiv (i, i)$ i) invokes a joint probability density densi function that complicates the complication of (8) because $\frac{1}{2}$

$$
(0) \qquad i
$$

$$
7, 13:
$$

mutual in-

$$
\Delta \mathbf{W} = \eta \left(\mathbf{I} - \psi(\mathbf{y}) \mathbf{y}^H \right) \mathbf{W}.
$$
 (9)

Liouville's theorem,
boundedness analyticity boundedness

$$
\begin{array}{cccc} \mathbf{9} \ , & 13 \ , & 14 \), & \end{array} \qquad , \qquad \qquad \begin{array}{c} \mathbf{3} \ , & \mathbf{4} \ , & \mathbf{5} \ , \\ & \mathbf{6} \ , & \mathbf{7} \ . \end{array}
$$

(6),
$$
(-2)
$$

\n7, 13:
\n
$$
\Delta W = \eta (I - \psi(y) y^H) W.
$$
 (9)
\n*Liouville's theorem,*
\n*boundedness*
\n*analyticity*
\n9, 13, 14;
\n
$$
\psi(\cdot) (\cdot, 7 - \cdot \cdot \cdot)
$$

\n9, 13, 14;
\n
$$
\psi(\cdot) (\cdot, 7 - \cdot \cdot \cdot)
$$

\n9, 13, 14;
\n
$$
\psi(\cdot) (\cdot, 7 - \cdot \cdot \cdot)
$$

\n10,
$$
\psi(\cdot) (\cdot, 7 - \cdot \cdot \cdot)
$$

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\psi(\cdot) (\cdot, 7 - \cdot \cdot \cdot)
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\psi(\cdot) (\cdot, 7 - \cdot \cdot \cdot)
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\psi(\cdot) (\cdot, 7 - \cdot \cdot \cdot)
$$

\n
$$
\psi(\cdot) (\cdot, 7 - \cdot \cdot \cdot)
$$

\n
$$
\mu(\cdot) \ni H(\cdot) \ni H(\
$$

$$
e_i = \begin{array}{cc} & - \\ i & \end{array},
$$

$$
H\begin{pmatrix} & | & i \\ i & \end{pmatrix}
$$

$$
H\left(\begin{array}{c|c|c|c} \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right) = H\left(\begin{array}{c|c} \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right) \equiv H(e_i) \tag{10}
$$

$$
H(\iota) \equiv H(\iota, \iota, \iota) \qquad \qquad -
$$

$$
H(\begin{array}{rcl}\ni & = & H(\begin{array}{rcl}i & | & i \end{array}) + H(\begin{array}{rcl}i &) \equiv H(-e_i) + H(\begin{array}{rcl}i &) \\
 & & i \end{array}) \\
 & = & H(\begin{array}{rcl}i & | & i \end{array}) + H(\begin{array}{rcl}i &) \equiv H(e_i) + H(\begin{array}{rcl}i &)\n\end{array})\n\end{array}
$$

$$
e_i \qquad \qquad - \qquad ; \qquad \ \, i \qquad \quad \ \,
$$

 $(6),$ applying the (2)

 $\begin{array}{c} \hline \end{array}$

$$
i k \t\t ; \t\t\t\t i j \neq 0.
$$
\n
$$
Q(\t i) > 0 \t\t R \t\t ; \t\t R
$$
\n
$$
\vdots \t\t R \t\t D. \Box
$$

Corollary 1: Q

$$
Q(\t_1 + \t_2) = Q(\t_1) + Q(\t_2) \tag{19}
$$

holds for two mutually independent circular complex vari- $\frac{1}{2}$ and $\frac{2}{3}$ is an effective contrast function $\frac{1}{2}$ is a contrast functio for separating instantaneous linear mixtures of independent

; it is equal complex sources; it is discrimination of $Q(|j| > 0$ for every independent source j . **Proof:** $\qquad \qquad j \tag{19}$

$$
Q \t (12)
$$
 (19)

$$
Q^{2}(\begin{array}{rcl}1+2) & = & Q^{2}(\begin{array}{rcl}1\end{array})+Q^{2}(\begin{array}{rcl}2\end{array})+2Q(\begin{array}{rcl}1)Q(\begin{array}{rcl}2\end{array})\\ &\geq & Q^{2}(\begin{array}{rcl}1\end{array})+Q^{2}(\begin{array}{rcl}2\end{array}),
$$

$$
Q(\t1) = Q(\t2) = 0.
$$
\n
$$
\begin{array}{cccc}\n2 & 1 & \text{sufficient} \\
& & \ddots & \vdots \\
& & & \ddots \\
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& & & & & & \ddots \\
& & & & & & & \ddots \\
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& & & & & & & & & & & & & & \ddots \\
& & & & & & & & & & & & & & \ddots\n\end{array}
$$

$$
not \qquad \text{cum}_k(\begin{array}{cc} 1+2 \end{array}) = \text{cum}_k(\begin{array}{cc} 1 \end{array}) + \text{cum}_k(\begin{array}{cc} 2 \end{array})
$$

$$
0, \theta_1, \theta_2 \in \mathbb{R}
$$
, $1 = |1|e^{j\theta_1}, 2 = |2|e^{j\theta_2},$

$$
1 + 2 = |1|e^{j\theta_1} + |2|e^{j\theta_2}
$$

\n
$$
= \sqrt{|1|^2 + |2|^2 + 2|1|} \cdot 2|\cos(\theta_1 - \theta_2)e^{j\theta},
$$

\n
$$
\theta = \arctan \frac{1}{1} \frac{n \theta_1 + 2}{\theta_1 + 2} \frac{n \theta_2}{\theta_2} \rightarrow \theta_1
$$

\n
$$
0 + \theta_2, \qquad |11 - |2| \le
$$

\n
$$
1 + 2| \le |1| + |2|.
$$

\n
$$
Theorem 3: \quad Q \qquad \qquad \frac{1}{2} = |e^{j\theta} (\theta \in \mathbb{R}),
$$

\n
$$
\overline{Q}(\theta) = Q(|\theta|) = \sqrt{Q^2(|\cos \theta| + Q^2(|\sin \theta|))}
$$

\n
$$
\qquad (2 - \theta)
$$

Proof: $1 = | 1|e^{j\theta_1}$ $a_2 = | 2|e^{j\theta_2}$
 $\vdots = | e^{j\theta} = 1 + 2,$ $\begin{array}{rcl} \n\cdot & = & | e^{j\theta} = 1 \\ \n\cdot & = & 1 + 2 \n\end{array}$ $_1$ + $\stackrel{+}{Q}$ 2 = $\frac{1}{1}$ 2 ; $\frac{1}{2}$; Q in real domain, we have Q $Q^2(|\cos\theta) \geq Q^2(|1|\cos\theta_1) + Q^2(|2|\cos\theta_2)$ $Q^2(|\sin\theta) \geq Q^2(|\sin\theta_1) + Q^2(|\sin\theta_2).$

 $Q^2() = Q^2(\mid \mid) = Q^2(\mid \mid \cos \theta) + Q^2(\mid \mid \sin \theta)$ $\geq Q^2(|1|) + Q^2(|2|) = Q^2(|1|) + Q^2(|2|),$ (20) $\frac{1}{5}$ and complete the proof. \sim 2-subadditivity. We can also prove 2-subadditivity. The canonical prove \Box $\hspace{1cm}$, if we can find a find functional of class II, while its real counterpart is superaded in \mathcal{I} , with $\frac{3}{2}$ functional of class II in complex domain. *Lemma 1:* $Q() (\in \mathbb{C})$
 \vdots $\tilde{\mathbf{z}} = [,],$ Q $\tilde{\mathbf{z}} = [\quad , \quad], \qquad Q$ also a function of class II in \mathcal{A} in real domain, with argument \mathcal{A} $\tilde{\mathbf{z}} \in \mathbb{R}^2;$ $\qquad \qquad$, $\qquad \qquad$ () $\qquad \qquad$
 $O(\alpha + \tilde{\mathbf{z}}) = O(\tilde{\mathbf{z}})$ $\qquad \qquad$ $\qquad \qquad$ $\qquad \qquad$ $Q(\boldsymbol{\alpha} + \tilde{\mathbf{z}}) = Q(\tilde{\mathbf{z}}) \; (\forall \boldsymbol{\alpha} \in \mathbb{R}^2);$ () $Q(\alpha \tilde{\mathbf{z}}) = |\alpha| Q(\tilde{\mathbf{z}}) \; (\forall \boldsymbol{\alpha} \in \mathbb{R})$ $Q(\alpha \tilde{\mathbf{z}}) = |\alpha| Q(\tilde{\mathbf{z}})$ ($\forall \alpha \in \mathbb{R}$). Proof: $|\alpha|e^{j0}$ (i.e., $\alpha =$). A 15 , contrast functions based on subadditive functional in the $\mathbf A$ setting; specifically, a theorem follows in the $\mathbf A$ below. *Theorem 4:* $Q \geq 2$, $\begin{array}{lll} \mathbf{z} & \mathbf{z} & \mathbf{z} \\ \mathbf{z} & \mathbf{z} & \mathbf{z} \\ \mathbf{z} & \mathbf{z} & \mathbf{z} \end{array}$ $-\sum_{i=1}^{n} Q^{k}(\alpha_i) - \sum_{i=1}^{n}$ $\sum_{i=1}^n Q^{2k}(-i)$ are contrast functions functions functions functions of \mathbb{R}^n for separating an instantaneous mixture of independent circular complex sources when the dem \mathbf{i} $\begin{align} &> 2 \quad \Omega \quad \{j\}, \end{align}$ **Proof:** $Q(y) = 0.$ **Proof:** The proof is considered in 15 $\begin{array}{ccc} \textbf{R} & = & \mathbf{W} \mathbf{A} & \mathbf{R} \\ \textbf{R} & \textbf{R}^2 & \textbf{A} \end{array}$ $Q,$ $Q^2(i) \leq \sum_{j=1}^n |j|^2 Q^2(j) \leq \sum_{j=1}^n |j|^2 Q^2(j)$ $\sum_{j=1}^{n} Q^{2} \begin{pmatrix} Q \\ j \end{pmatrix}$
2, $\begin{array}{cc} \binom{n}{j+1} & Q^2(j) & |j| \leq \sum_{k=1}^n \frac{1}{j+k} & |j|^2 = 1; \\ O^k(j) & \leq \sum_{k=1}^n \frac{1}{k} & |j|^2 \leq \sum_{k=1}^n \frac{1}{j+k} \end{array}$ \overline{Q} , $Q^k(\iota) \leq \sum_{i=1}^n |\iota_j|^k Q^k(\iota_j) \leq \sum_{j=1}^n Q^k(\iota_j)$
 $-\sum_{i=1}^n Q^k(\iota_i) \geq -\sum_{j=1}^n Q^k(\iota_j)$,

 $\mathcal{S}_{\mathcal{S}}$ summing up the above two inequalities and because of scales and because of scales and because of scales and

equivariance, we obtain

 $-\sum_{i=1}^n Q^k(-_i)$

$$
\mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{A}\mathbf{s} \equiv \tilde{\mathbf{A}}\mathbf{s}, \qquad \begin{aligned} \mathbf{z} &= \mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{x} = \\ \tilde{\mathbf{A}} &= \mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{A} \\ \mathbb{E}[\mathbf{z}\mathbf{z}^H] &= \tilde{\mathbf{A}}\mathbb{E}[\mathbf{s}\mathbf{s}^H]\tilde{\mathbf{A}}^H = \mathbf{I}; \\ \tilde{\mathbf{A}} &= \mathbb{E}[\mathbf{s}\mathbf{s}^H] = \mathbf{I}. \end{aligned}
$$

B. Examples of Contrast Functions

1) Range Function: In [15] and [21], range function was used as a contrast function for real-valued ICA, we may also generalize its use to complex domain. Assume all complex variables are distributed within bounded support

 \ldots , the probability that the probability that the probability that the complex variables is the complex variables is the complex variables in \ldots inside the bounded or finite support circle is non-negative, ϵ and zero elsewhere, the range of a complex \mathcal{A}

 $\sum_{i=1}^n Q^k(-i)$ is a contrast of the proof the pr is omitted due to lack of space. *Remark:*
 $\mathbb{F}[xx^H] =$
 $\mathbb{F}[xx^H] =$ $\mathbf{A} \mathbb{E}[\mathbf{s} \mathbf{s}^H] \mathbf{A}^H$ = $\mathbf{I} \mathbf{I} \mathbf{\Sigma} \mathbf{I}^H$ $\qquad \qquad \nabla$ $\qquad \nabla$ $\qquad \nabla$ $\mathbf{A}\mathbb{E}[\mathbf{s}\mathbf{s}^H]\mathbf{A}^H = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^H, \hspace{1cm} \Sigma$ with nonnegative real-valued entries, and **U** is a unitary

$$
R() , \t R() = d , \t R() \geq 0 \mid | -_0 | \leq d \} \t (21)
$$

 $\begin{array}{ccc} 0 & & \\ & d \subset \mathbb{D}^+ \end{array}$ o
 $d \in \mathbb{R}^+$ *Theorem 5:* ables 1 are independent, then the independent, then the independent, then the independent satisfies the satis \sim

Proof:

$$
R(\begin{array}{cc} 1+2 \end{array}) = R(\begin{array}{cc} 1 \end{array}) + R(\begin{array}{cc} 2). & (22)
$$

 $\begin{array}{ccc} | & 1 | + | & 2| & \overline{} \\ \hline \text{Corollary 2:} & = \alpha_1 + \beta_2, & 1 & 2 \end{array}$ $=\alpha_1 + \beta_2$, independent circular complex bounded random variables, and $\alpha, \beta \in \mathbb{C}$,

$$
R(\) = |\alpha| \cdot R(\ _1) + |\beta| \cdot R(\ _2). \tag{23}
$$

 $\alpha_1(\begin{array}{cc} \beta_2 \end{array})$ is $\alpha_2(\begin{array}{cc} \beta_3 \end{array})$ is just $\alpha_3(\begin{array}{cc} \beta_4 \end{array})$ $_1$ ($_2$), α (β).

$$
Q(\iota) = R(\iota), \qquad (17)
$$

$$
J(\mathbf{W}) = \sum_{i=1}^{n} \log \left(\sum_{j=1}^{n} |\iota_j| R(\iota_j) \right) - \log |\mathbf{W}|. \quad (24)
$$

 $\mathcal{M}_{\mathcal{A}}$ used for finite circular complex sources.

Definition 6: Let be a circular complex random variable

$$
\begin{array}{ccc}\n & 1 & | & \leq +\infty \\
 & \int_0^d & (| \ |)d | & | \approx 1 & (0 < d < +\infty) \\
 & & & d & & \\
 & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
\end{array}
$$

2) Shannon Entropy Function: A information-theoretic contrast function is the marginal

$$
H(\gamma_i).
$$

 $\overline{Q}(\,i) = Q(|\,i|) = \exp(H(|\,i|)),$ (25)

entropy power inequality, (25) is a superadditive functional of class II in real domain.

From Theorem 3 we can also construct a superadditive

$$
(25) \qquad (17) \qquad (6)
$$

$$
\{i\}, \qquad \{|\ i|\}
$$

$$
\begin{array}{c}\n16,17\n\end{array}
$$

3) Rényi Entropy Function:

$$
(0 < \in \mathbb{R}):
$$
\n
$$
H_k(\ _i) = \frac{1}{1 -} \log \Big(\int (\ _i)^k d_i \Big). \tag{26}
$$
\n
$$
\to 1 \qquad , \qquad = 2,
$$

2 extension entropy

$$
H_2(a_i) = -\log\Big(\int (a_i)^2 d_i\Big).
$$

By *Jensen inequality*, $H_2(u) \leq H(u)$. $\mathcal{L}_{\mathcal{A}}$

$$
H_k(\begin{array}{ccc} & H_k(\begin{array}{c} \\ i \end{array}) \geq H_r(\begin{array}{c} \\ i \end{array}) & > \quad .
$$
\n
$$
g(\begin{array}{c} \\ j \end{array})\},
$$
\n
$$
g(\begin{array}{c} \\ j \end{array})\},
$$
\n
$$
g(\begin{array}{c} \\ j \end{array})\},
$$

$$
\begin{array}{ll}\n\exp(H_2(\)),\\
\textbf{Proof:} \qquad,\qquad H_2(\)=H_2(\ +\alpha)\\
\alpha.\qquad,\qquad\end{array}
$$

$$
H_2(\alpha) = -\log \left(\int g(|\alpha|)^2 d|\alpha| \right)
$$

= $-\log \left(\int g(|\alpha|)^2 d|\alpha| \right) + \log |\alpha|.$

$$
\exp(H_2(\alpha)) = |\alpha| \exp(H_2(\alpha)).
$$

\n
$$
\exp(H_2(\alpha)) = \frac{\overline{Q}(\alpha)}{\overline{Q}(\alpha)} = Q(|\alpha|) = \exp(H_2(|\alpha|))) = \frac{\overline{Q}(\alpha)}{\int_{\alpha}^{\alpha} P(|y_i|^2)^2 dy_i},
$$

$$
26.27 ; \qquad \overline{Q} \qquad \qquad \{ | \ _i | \},
$$

4) Fisher Information Function:

the functional of $\mathcal{F}_\mathcal{F}$ information $\mathcal{F}_\mathcal{F}$

$$
\mathbf{G} = \mathbb{E}\big[\psi(\)^2\big] = \mathbb{E}\Big[\Big(\frac{d\log\ (\)}{d}\Big)^2\Big],\tag{27}
$$

$$
\psi(\) = \frac{d - p(\)}{d} = \frac{p'(\)}{p(\)} \nQ(\) = \ \mathbf{G}^{-1/2},
$$

 15 .

$$
\overline{Q}(\) = \mathbb{E}\Big[\Big(\frac{\prime(\vert \ \vert)}{\vert \ \vert}\Big)^2\Big]^{-1/2}.
$$

$$
\psi_k(\) = (\)^T \qquad , \qquad j/T \qquad T \ /
$$

$$
Q(\) = \ \ (\Sigma \)^{1/2} \equiv \ \left(\text{cov}[\tilde{\mathbf{z}}]\right)^{1/2},
$$

\n
$$
Q(\omega \) = \ \left(\text{cov}[\tilde{\mathbf{z}}']\right)^{1/2} = \ \left(\text{cov}[\Omega \tilde{\mathbf{z}}]\right)^{1/2}
$$

\n
$$
= \ \left(\Omega \text{cov}[\tilde{\mathbf{z}}]\right)^{1/2} = \ \left(\Omega\right)^{1/2} \ \left(\text{cov}[\tilde{\mathbf{z}}]\right)^{1/2}
$$

\n
$$
= \ \left(\omega^2 + \omega^2\right)^{1/2} \ \ (\Sigma \)^{1/2} = |\omega| \ \ (\Sigma \)^{1/2}.
$$

\n
$$
Q(\)
$$

\n
$$
\vdots
$$

\n
$$
Q(\)
$$

complex sources. Obviously, there are still many important many importa theoretical issues that require functions \mathbf{A} numbers \mathbf{A} ber of them are listed here. • As A , spurit A , spurit in the real minimal minim exist for information-theoretic criteria (such as mutually $($ information or negentropy) as well as cumulant-based (extending as kurtosis) $22-24$, it is interesting to interesting to interest interesting to interest interesting to interest in A . examine such phenomena in complex ICA. • Typical optimization algorithms for \mathbf{A}_{\perp} use local grapdient search, with orientation 25 (**W** is a unitary matrix). The $\mathcal{U}(\cdot)$, $\mathcal{U}(\cdot)$, $\mathcal{U}(\) ,$ which is a Lie $\mathcal{U}(\) ,$ group of dimension $\mathcal{Z}_\mathbf{S}$ such a special structure may be special structure may be special structure may be require special design of the optimization algorithm. • Although we only discuss simultaneous simultaneous separation approximation approximation approximation approximation approximation $\mathbf A$ in this paper, it would be also be als interesting to investigate the contrast function for the sequential extraction (i.e., definition) approach, which is (x, y, \ldots, y) can be viewed as a degenerate case of the former ap- \mathbf{p} both cases, stability and algorithm analysis of the algo

is important. VII. CONCLUDING EMARKS \overline{A} problems of \overline{A} , such as separations, such as separation of $\mathcal{T},$ 13 , speech processing in frequency domain 14 , communi-

 $5, 12$. A $\mathbf A$ [6]-[14]. However, the theory of complex ICA is still not as well understood as its real counterpart, especially for simulations \mathcal{A} taneous separation. Essentially, designing effective contrasting ef functions is the key to the solution of \mathcal{A} contrast functions have been studied for the circular complex sources, and we overview and extend some established work and derive several new results. The current paper is purely $\frac{1}{2}$ of efficient algorithms are the topics of future investigation.

A \bar{X} For a real-valued random variable (which can be either $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ number), we may obtain its up-to-fourth-order Gram-Charlier its up-to-fourth-order Gram-Charlier in \sim $() \approx \mathcal{N} () \left(1 + \frac{\kappa^3}{6} \mathcal{H}_3() + \frac{\kappa^4}{2} \mathcal{H}_4() \right)$