Applications of Mixed Hodge Module Theory
混合霍奇模的应用

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Saito's Theory

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Moral: mixed Hodge module theory over C parallels mixed l-adic sheaf theory over a eld with characteristic $p > 0$.

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2. If $X = pt$, then $MHM(X) = MHS$ the category of mixed Hodge structures and rat takes a mixed Hodge structure to its underlying Q vector space.
3. It lifts six-functor formalism in $D^b(\mathrm{Perv}_Q(X)) = D^b_c(X)$ to $D^b(MHM(X))$, and other useful functors (duality functor, nearby and vanishing cycles, complex-conjugation functor).

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  5. (v) Zucker: global results (L^2-cohomology).

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  - Koszul duality: [AK14].
  - Categorical action of $\mathfrak{sl}_2$: [CDK16].

Definition

Let $\lambda \in \mathfrak{t}^*$ be arbitrary. Then the Verma module $V(\lambda)$ admits a decreasing filtration $V(\lambda)^\bullet$ by submodules, such that

(i) $V(\lambda)^i = 0$ for all sufficiently large $i >> 0$.

(ii) Vmmmdmmmm[]dmdmmm[]ddmm

Construction

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- Given a deformation direction $\gamma \in \mathfrak{t}^*$, one can consider the deformed Verma module $V_{\mathbb{C}[s]}(\lambda)$ which is a $(\mathfrak{g}, \mathbb{C}[s])$-bimodule generated by a highest weight vector v_λ satisfying
$$h \cdot v_\lambda = (\lambda(h) + s\gamma(h))v_\lambda, \forall h \in \mathfrak{t}.$$
- Contravariant form: $\mathbb{C}[z]$-bilinear contra-variant form on the deformed Verma module $V_{\mathbb{C}[z]}(\lambda)$ is non-degenerate, which specializes at $z = 0$ to the contra-variant form on $V(\lambda)$.

Construction

- Filtration: On $V_{\mathbb{C}[z]}(\lambda)$, one has a filtration by order of vanishing of the form, i.e.

$$V_{\mathbb{C}[z]}(\lambda)^i := \sum_{\nu\in\Gamma} V_{\lambda_s-\nu}(i),$$

where Γ is the set of $\mathbb{Z}^+-$linear combinations of simple roots and

$$V_{\lambda_s-\nu}(i) := \{v \in V_{\lambda_s-\nu} : (v, V_{\lambda_s-\nu}) \subset s^i\mathbb{C}[s]\}.$$

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- The theorem above is actually the deformation in the direction $\lambda = \rho$.

Jantzen Conjecture

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  1. Certain canonical maps (e.g. embeddings $V(\mu) \hookrightarrow V(\lambda)$) are strict for Jantzen filtrations.
  2. The Jantzen filtration coincides with the socle filtration.
- More precisely, let $\chi_1, \chi_2 \in \mathfrak{t}^*_{\mathbb{Q}}$ be regular weights such that $V(\chi_1) \subseteq V(\chi_2)$, which means that there exists some dominant weight χ such that $\chi_i = w_i\chi$ with $w_i \in W^\chi := \{w \in W : w \cdot \chi - \chi \in \mathfrak{h}^*_{\mathbb{Z}}\}$ and $w_1 \leq w_2$. Then we have the following relation: $V(\chi_1)^i = V(\chi_1) \cap V(\chi_2)^{i+l(w_2)-l(w_1)}$.

Remarks

- The Jantzen filtration is a very useful tool in the representation theory of Lie algebras.
  - The structure of Verma modules over a rank two simple Lie algebra (or a Kac-Moody algebra) is completely determined by means of the Jantzen filtration [Jan79].

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  - The structure of Verma modules over a rank two simple Lie algebra (or a Kac-Moody algebra) is completely determined by means of the Jantzen filtration [Jan79].
  - Gabber and Joseph [GJ81] showed that (1) implies the Kazhdan-Lusztig conjectures on multiplicities of simple modules in Verma modules.
- Building on the work of Gabber and Joseph, Barbasch [Bar83] showed that (1) implies (2) in a purely algebraic approach.

Bernstein-Beilinson's Proof

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- Part (1) follows from weight formalism in the theory of mixed $l-$adic sheaves.
- Part (2) is proved by a pointwise purity argument.

Dictionary

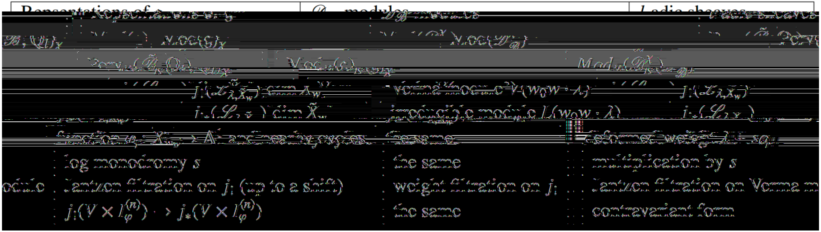

图 1: Dictionary for Jantzen conjecture

Advantages

- To use [BBDG18], one needs the condition "of geometric origin". One can only prove Verma modules with rational characters are of geometric origin, [BB93, Lemma 2.6.5]
- However, we don't need this condition if we use Mixed Hodge module theory.
- We can remember the polarization via mixed Hodge Module theory.

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- However, we don't need this condition if we use Mixed Hodge module theory.
- We can remember the polarization via mixed Hodge Module theory.
- We use mixed Hodge modules with $\mathbb{C}$-coefficients.

Main Theorem

Fix $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$, a K-orbit $Q \subseteq B$ and an irreducible K-equivariant λ-twisted flat bundle $\mathcal{V}$ on Q. Let S be a polarization of $\mathcal{V}$. Then for all n, $Gr^J_{-n} j_! \mathcal{V}$ is a pure Hodge module of weight $d-n$, and the form

$$s^{-n} Gr^J_{-n}(S) : Gr^J_{-n} j_! \mathcal{V} \xrightarrow{\cong} (Gr^J_{-n} j_! \mathcal{V})^h(-d+n)$$

is a polarization, where $(-)^h$ denotes the Hermitian dual and $d = \dim H + \dim Q$.

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- $\overline{\mathscr{M}}$ to denote the complex conjugation of $\mathscr{D}$-module $\mathscr{M}$,

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Now a polarized complex mixed Hodge module $\mathscr{M}$ consists of the following data:

- a triple $(\mathscr{M}, F_\bullet\mathscr{M}, W_\bullet\mathscr{M})$, where $\mathscr{M}$ is a regular holonomic $\mathscr{D}_X$ module, Hodge filtration $F_\bullet$ is a good filtration by $\mathscr{O}_X$–modules, and weight filtration $W_\bullet$ is a filtration by regular holonomic $\mathscr{D}_X$-submodules;

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- a triple $(\mathscr{M}', F_\bullet\mathscr{M}', W_\bullet\mathscr{M}')$, similar as above;
- a perfect sesquilinear paring $\mathfrak{s} : \mathscr{M} \otimes \overline{\mathscr{M}'} \to \mathrm{D}b_X$ compatible with $W_\bullet$ (i.e. induces an isomorphism $\mathscr{M} \cong (\mathscr{M}')^h$ of underlying $\mathscr{D}_X$-modules);

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- Recall that $\tilde{\mathscr{B}} \to \mathscr{B}$ is a H–torsor and $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$. We define the category of λ–twisted mixed Hodge Modules on $\mathscr{B}$, denoted by $MHM_\lambda(\mathscr{B})$ to be the full subcategory of $MHM(\tilde{\mathscr{B}})$, consisting of all the objects whose underlying $\mathscr{D}$-module is the pull-back of a λ–twisted $\mathscr{D}_{\mathscr{B}}$-module on $\mathscr{B}$.

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- If $\lambda \notin \mathfrak{h}_{\mathbb{R}}^*$, then the category $MHM_\lambda(\mathscr{B})$ (defined in a similar way) must be zero (claimed in [DV22]).

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Jantzen Filtration on $\mathscr{D}$-modules

- Fix $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ and a K-orbit $Q \subseteq B$.By [BB93, Lemma 3.5.2], there exists $\varphi \in \mathbb{X}^*(H)$ and a K-invariant section $f_\varphi \in H^0(Q, L^\varphi)$ such that $f_\varphi^{-1}(0) \cap Q^- = \partial Q$.
- Now we form the $(\lambda + s\varphi)$–twisted flat bundle $\mathcal{V}_{s\varphi}$ on Q.
- As $\mathcal{O}_Q$ modules, $\mathcal{V}_{s\varphi}$ is the same as $\mathcal{V}$ (we use $f_\varphi^s m \in \mathcal{V}_{s\varphi}$ to denote the corresponding $m \in \mathcal{V}$)

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- but the $\mathscr{D}_Q-$module structure is given by

$$\partial f_\varphi^s m = f_\varphi^s(\partial m + s\frac{\partial f_\varphi}{f_\varphi}m). \tag{1}$$

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- We use $\tilde{Q}$ to denote the preimage of Q under $\tilde{\mathscr{B}} \to \mathscr{B}$.
- Consider the family of tautological morphisms $j_!\mathcal{V}_{s\varphi} \to j_*\mathcal{V}_{s\varphi}$. In order to consider its behavior near $s = 0$ we pass to the formal completion $j_!\mathcal{V}_{s\varphi}[[s]] \to j_*\mathcal{V}_{s\varphi}[[s]]$,
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- Our assumption on φ implies that this map is an isomorphism after inverting s.
- It induces Jantzen filtrations $J_\bullet$ on the domain and codomain defined by $J_n j_!\mathcal{V} = (j_!\mathcal{V}_{s\varphi}[[s]] \cap s^{-n} j_*\mathcal{V}_{s\varphi}[[s]])/(s)$, and $J_n j_!\mathcal{V} = (s^{-n} j_!\mathcal{V}_{s\varphi}[[s]] \cap j_*\mathcal{V}_{s\varphi}[[s]])/(s)$, and isomorphisms $s^n : Gr^J_n j_*\mathcal{V} \xrightarrow{\cong} Gr^J_{-n} j_!\mathcal{V}$.

Proof

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- Step 2: relate the graded pieces of Jantzen filtration to Beilinson's functors (to imitate [BB93]).
- Step 3: verify $s^{-n} Gr^J_{-n} S$ coincides with the polarization on nearby cycles.

Ideas

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- Idea: "Lift" results on $\mathscr{D}-$modules to results on complex mixed Hodge modules.
- Step 1 is standard.
- Step 2 is almost repeating word for word [BB93].
- The key idea is to lift the comparison theorem $\pi_f^1 \cong {}^p\psi_f^{un}$ in $\mathscr{D}-$modules setting ([Bei87]) to mixed Hodge module setting. In particualr, the deformation parameter s corresponds to the nilpotent operator s on π_f^1.

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- $$H^i(Z, \mathrm{gr}_p^F \mathrm{DR}(M) \otimes \mathscr{L}) = 0, \text{ for } i > 0 \text{ and } p \in \mathbb{Z},$$
  $$H^i(Z, \mathrm{gr}_p^F \mathrm{DR}(M) \otimes \mathscr{L}^{-1}) = 0, \text{ for } i < 0 \text{ and } p \in \mathbb{Z}.$$

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  - $F_p \mathrm{DR}_X(\mathscr{O}_X) = [F_p \mathscr{O}_X \to \Omega_X^1 \otimes F_{p+1} \mathscr{O}_X \to \cdots \to \Omega_X^n \otimes F_{p+n} \mathscr{O}_X][n]$,

Corollary: Kodaira's Vanishing Theorem

- If X is a projective variety of complex dimension n, $\mathscr{L}$ any ample line bundle on X, and ω_M is the canonical line bundle,
- then

$$H^q(X, \omega_X \otimes \mathscr{L}) = 0, \;\; q > 0,$$
$$H^q(X, \mathscr{L}^{\otimes -1}) = 0, \;\; q < n.$$

- Proof: consider $\mathbb{Q}_X^H[n]$. Note that
  - $F_p\mathrm{DR}_X(\mathscr{O}_X) = [F_p\mathscr{O}_X \to \Omega_X^1 \otimes F_{p+1}\mathscr{O}_X \to \cdots \to \Omega_X^n \otimes F_{p+n}\mathscr{O}_X][n]$,
  - $\mathrm{gr}_p^F\mathrm{DR}_X(\mathscr{O}_X) = \begin{cases} \Omega_X^{-p} \otimes \mathscr{O}_X[n+p], & \text{if } -n \le p \le 0, \\ 0, & \text{otherwise.} \end{cases}$

Proof

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- Step 5: compare with original complex.
- Step 6: connect the dots.

Step 2: duality argument

- Theorem: Let $\mathscr{M} \in HM(X,\omega)$ be a polarizable Hodge module on an n-dimensional complex manifold X. Then any polarization on $\mathscr{M}$ induces an isomorphism

$$R\mathscr{H}om_{\mathscr{O}_X}(\mathrm{gr}_p^F \mathrm{DR}(\mathscr{M}), \omega_X[n]) \cong \mathrm{gr}^F_{-p-\omega}\mathrm{DR}(\mathscr{M}).$$

- Note that by Serre duality,

$$R^i\mathscr{H}om_{\mathscr{O}_X}(\mathrm{gr}_p^F \mathrm{DR}(\mathscr{M}), \omega_X[n]) \cong R^{i+n}\mathscr{H}om_{\mathscr{O}_X}(\mathrm{gr}_p^F \mathrm{DR}(\mathscr{M}), \omega_X) \cong H$$

- Combining the results above, we obtain

$$H^{-i}(X, \mathrm{gr}_p^F \mathrm{DR}(\mathscr{M}))^* \cong H^i(X, \mathrm{gr}^F_{-p-\omega}\mathrm{DR}(\mathscr{M})).$$

Step 4: strictness of MHM to get vanihsing of morphism

Now using strictness of pushforward to a point, we have E_1-degeneration of the spectral sequence

$$E_1 = H^{p+q}(Y, \mathrm{gr}^F_{-p}\mathscr{M}_Y) \Rightarrow H^{p+q}(Y, \mathscr{M}_Y)$$

and hence we obtain that the morphism

$$H^i(Y, \mathrm{gr}^F_p\mathscr{M}_Y) \to H^{i+1}(Y, \mathrm{gr}^F_{p-1}\mathscr{M}_Y)$$

is zero map for all $i \in \mathbb{N}$ and $p \in \mathbb{Z}$.

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Thanks!