Grothendieck's absolute purity conjecture
Motivic homotopy theory
The fundamental class
Absolute purity in motivic homotopy theory

Absolute purity in motivic homotopy theory

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joint work with F. Deglise, J. Fasel and A. Khan

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The absolute purity conjecture

Grothendieck's **absolute (cohomological) purity conjecture**
(SGA5, Expose I 3.1.4) is the following statement: if $i : Z \to X$ is
a closed immersion between noetherian regular schemes of pure
codimension c, $n \in \mathcal{O}(X)$ and $\Lambda = \mathbb{Z}/n\mathbb{Z}$, then the etale
cohomology sheaf supported in Z with values in Λ can be
computed as

$$\mathcal{H}_Z^q(X_{\text{et}}; \Lambda) = \begin{cases} i_* \Lambda_Z(-c) & \text{if } q = 2c \\ 0 & \text{else} \end{cases}$$

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In other words, $i^! \ _X = \ _Z(-c)[-2c]$.

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This conjecture has been solved by Gabber.

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A short history of the proof

- SGA4: case where both X and Z are smooth over a field

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  Based on Thomason's method + rigidity for algebraic K-theory

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Importance of the absolute purity conjecture

The absolute purity property, together with resolution of singularities, is frequently used in cohomological studies of schemes:

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The absolute purity property, together with resolution of singularities, is frequently used in cohomological studies of schemes:

- Show that the six functors on the derived category of etale sheaves preserve constructible objects.
- Prove the *Grothendieck-Verdier local duality*:

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  S a regular scheme, $n \in \mathcal{O}(S)$ and $\ = \mathbb{Z}=n\mathbb{Z}$, $f : X \to S$ a separated morphism of nite type, then $f^! \ _S$ is a dualizing object, i.e. $\mathbb{D}_{X/S} := R\underline{Hom}(\cdot; f^! \ _S)$ satis es $D \circ D = \text{Id}$.

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- Show that the six functors on the derived category of etale sheaves preserve constructible objects.

- Prove the *Grothendieck-Verdier local duality*:
  S a regular scheme, $n \in \mathcal{O}(S)$ and $\Lambda = \mathbb{Z}/n\mathbb{Z}$, $f : X \to S$ a separated morphism of finite type, then $f^! \Lambda_S$ is a dualizing object, i.e. $\mathbb{D}_{X/S} := R\underline{Hom}(\cdot, f^! \Lambda_S)$ satisfies $D \circ D = \mathrm{Id}$.

- Construct Gysin morphisms and establish intersection theory.

- Study the coniveau spectral sequence.

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- Our work: study absolute purity in the framework of motivic homotopy theory.

- Main result: the absolute purity in motivic homotopy theory is satis ed with rational coe cients in mixed characteristic.

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Motivic homotopy theory

- The motivic homotopy theory or $\mathbb{A}^1$-homotopy theory is introduced by Morel and Voevodsky (1998) as a framework to study cohomology theories in algebraic geometry, by importing tools from algebraic topology

- Idea: use the affine line $\mathbb{A}^1$ as a substitute of the unit interval to get an algebraic version of the homotopy theory

- Can be used to study cohomology theories such as algebraic K-theory, Chow groups (motivic cohomology) and many others

- Advantage: has many a lot of structures coming from both topological and algebraic geometrical sides

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Aspects of applications in various domains

- Part of Voevodsky's proof of the Bloch-Kato conjecture uses the classi cation of cohomological operations that can be studied by means of motivic homotopy theory

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- $\mathbb{A}^1$-enumerative geometry (Hoyois, Levine, Kass-Wickelgren)

- Non-commutative geometry and singularity categories (Tabuada, Blanc-Robalo-Toen-Vezzosi)

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Some topological background

- A **spectrum** $\mathbb{E}$ is a sequence $(E_n)_{n \in \mathbb{N}}$ of pointed spaces (e.g. CW-complexes or simplicial sets) together with continuous maps $\sigma_n : S^1 \wedge E_n \to E_{n+1}$ called **suspension maps**

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$$\pi_n(E) = \operatorname*{colim}_i \pi_{n+i}(E_i)$$

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- The theory stems from the Freudenthal suspension theorem: if $E_i = X \wedge S^i$ for some $X \in Top$ (i.e. E is the **suspension spectrum** of X), then the sequence $i \mapsto \pi_{n+i}(E_i)$ stabilizes

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Some topological background

- A **spectrum** $\mathbb{E}$ is a sequence $(E_n)_{n \in 2\mathbb{N}}$ of pointed spaces (e.g. CW-complexes or simplicial sets) together with continuous maps $\sigma_n : S^1 \wedge E_n \to E_{n+1}$ called **suspension maps**
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Stable homotopy category

- The (topological) stable homotopy category **SH**$_{top}$ is de ned from spectra by inverting stable weak equivalences

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$$\mathbb{E}^n(X) = [X; \mathbb{E} \wedge S^n]_{\mathbf{SH}_{top}}$$

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- Examples: Suspension spectra $\Sigma^\infty X$ for $X \in Top_*$, in particular sphere spectrum $\mathbb{S}$; HA Eilenberg-Mac Lane spectrum for a ring A; MU complex cobordism spectrum
- From an ∞-categorical point of view, the category of spectra is the stabilization of the category of spaces, and is the universal stable (triangulated) category

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The unstable motivic homotopy category

- For any scheme S, a **motivic space** is a presheaf of simplicial sets over the category of smooth S-schemes Sm_S

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- For any scheme S, a **motivic space** is a presheaf of simplicial sets over the category of smooth S-schemes Sm_S

- The **(pointed) unstable motivic homotopy category $\mathbf{H}(S)$** ($\mathbf{H}(S)$) is obtained from (pointed) motivic spaces by localizing with respect to the Nisnevich topology and projections of the form $Y \times \mathbb{A}^1 \to Y$

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- Bigraded $\mathbb{A}^1$-**homotopy sheaves**: for $X \in \mathbf{H}_\bullet(S)$, $\mathbb{A}^1_{a,b}(X)$ is the Nisnevich sheaf on Sm_S associated to the presheaf

$$U \mapsto [U \wedge S^{a-b} \wedge \mathbb{G}_m^b, X]_{\mathbf{H}_\bullet(S)}$$

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- For any scheme S, a **motivic spectrum** or $\mathbb{P}^1$**-spectrum** is a sequence $\mathbb{E} = (E_n)_{n \geqslant 0}$ of pointed motivic spaces together with morphisms $\quad_n : \mathbb{P}^1 \wedge E_n \to E_{n+1}$

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- The **stable motivic homotopy category SH**(S) is de ned from $\mathbb{P}^1$-spectra by inverting stable motivic weak equivalences

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- **SH**(S) is the universal stable ∞-category which satis es Nisnevich descent and $\mathbb{A}^1$-invariance (Robalo, Drew-Gallauer)

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Motivic spectra

Every object in **SH**(S) represents a bigraded cohomology theory

$$\mathbb{E}^{p,q}(U) = [U, (S^1)^{\wedge(p-q)} \wedge (\mathbb{G}_m)^{\wedge q} \wedge \mathbb{E}]_{\mathbf{SH}(S)}$$

- Motivic Eilenberg-Mac Lane spectrum **H**$\mathbb{Z}$, represents motivic cohomology (extend Chow groups for smooth schemes)

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- Algebraic cobordism spectrum **MGL**, represents algebraic cobordism (Levine-Morel)
- Hermitian K-theory spectrum **KQ** represents higher Grothendieck-Witt groups (Schlichting, Panin-Walter, Hornbostel)

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Motivic spectra

Every object in **SH**(S) represents a bigraded cohomology theory

$$\mathbb{E}^{p,q}(U) = [U_+, (S^1)^{\wedge (p-q)} \wedge (\mathbb{G}_m)^{\wedge q} \wedge \mathbb{E}]_{\mathbf{SH}(S)}$$

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- Milnor-Witt spectrum **H**$_{MW}\mathbb{Z}$ represents Milnor-Witt motivic cohomology/higher Chow-Witt groups (Deglise-Fasel)

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The sphere spectrum

- The sphere spectrum $\mathbb{1}_S = {}^1_{\mathbb{P}^1} S_+$ is the unit object for the monoidal structure on **SH**(S) de ned by $\otimes = \wedge$

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- The 1-line is also computed (Röndigs-Spitzweck-Østvaer):

$$0 \to K_2^M{}_n=24 \to {}_{n+1,n}(\mathbb{1}_k) \to {}_{n+1,n}f_0(\mathbf{KQ})$$

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The six functors formalism

- Originates from Grothendieck's theory for l-adic sheaves (SGA4), and worked out in the motivic setting by Ayoub and Cisinski-Deglise

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The six functors formalism

- Originates from Grothendieck's theory for l-adic sheaves (SGA4), and worked out in the motivic setting by Ayoub and Cisinski-Deglise
- For any morphism of schemes $f : X \to Y$, there is a pair of adjoint functors

$$f^* : \mathbf{SH}(Y) \rightleftarrows \mathbf{SH}(X) : f_*$$

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- They satisfy formal properties axiomatizing important theorems such as duality, base change and localization.

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Thom spaces and relative purity

- If $V \to X$ is a vector bundle, then the **Thom space** $Th_X(V) \in \mathbf{H}(X)$ is the pointed motivic space $V/V - 0$

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- Relative purity (Ayoub): $f : X \to Y$ smooth morphism with tangent bundle T_f, then $f^! \simeq Th(T_f) \otimes f$

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- Relative purity (Ayoub): $f : X \to Y$ smooth morphism with tangent bundle T_f, then $f^! \simeq Th(T_f) \otimes f$
- In the presence of an *orientation*, we recover the usual relative purity

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Orientations

- An **absolute motivic spectrum** is the data of $\mathbb{E}_X \in \mathbf{SH}(X)$ for every scheme X, together with natural isomorphisms $f^*\mathbb{E}_X \simeq \mathbb{E}_Y$ for every morphism $f : Y \to X$

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- Examples: $\mathbb{1}$, $\mathbf{H}\mathbb{Z}$, $\mathbf{KGL}$, $\mathbf{MGL}$, $\mathbf{KQ}$, $\mathbf{H}_{MW}\mathbb{Z}$

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- An **orientation** of $\mathbb{E}$ is the data of isomorphisms $E_X \otimes Th_X(V) \simeq E_X(r)[2r]$ for all vector bundles $V \to X$ of rank r, which is compatible with pullbacks and products

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- Non-examples: $\mathbb{1}$, $\mathbf{KQ}$, $\mathbf{H}_{MW}\mathbb{Z}$

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Orientations and fundamental classes

- The algebraic cobordism spectrum **MGL** is the universal oriented absolute spectrum

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- With an orientation, we have an associated formal group law, as well as many extra properties such as projective bundle formula or double point formula (Levine-Pandharipande)

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- For oriented spectra, Deglise de ned fundamental classes using Chern classes

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Bivariant groups

- For $f : X \to S$ be a separated morphism of finite type, $v \in K_0(X)$ and $\mathbb{E} \in \mathbf{SH}(S)$, define the $\mathbb{E}$-**bivariant groups** (or **Borel-Moore $\mathbb{E}$-homology**) as

$$\mathbb{E}_n(X/S, v) = [f_! Th(v)[n], \mathbb{E}]_{\mathbf{SH}(S)}$$

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- If S is a field and $\mathbb{E} = \mathbf{H}\mathbb{Z}$, then $\mathbb{E}_i(X/S; v) = CH_r(X; i)$ are the higher Chow groups, where r is the virtual rank of v

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Functoriality of bivariant groups

- Base change:

$$
\begin{array}{ccc}
Y & \xrightarrow{\quad q \quad} & X \\
{\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\
T & \xrightarrow{\quad p \quad} & S
\end{array}
$$

$$\Delta^* : \mathbb{E}_n(T=S; v) \to \mathbb{E}_n(Y=X; g^* v)$$

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$$f_* : \mathbb{E}_n(X=S; f^{!}v) \to \mathbb{E}_n(Y=S; v)$$

- Product: if $\mathbb{E}$ has a ring structure, $X \xrightarrow{f} Y \xrightarrow{g} S$

$$\mathbb{E}_m(X=Y; w) \otimes \mathbb{E}_n(Y=S; v) \to \mathbb{E}_{m+n}(X=S; w + f^{!}v)$$

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The fundamental class (Deglise-J.-Khan)

- We say that a morphism of schemes $f : X \to Y$ is *local complete intersection* (lci) if it factors as a regular closed immersion followed by a smooth morphism

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- We say that a morphism of schemes $f : X \to Y$ is *local complete intersection* (lci) if it factors as a regular closed immersion followed by a smooth morphism

- To such a morphism is associated a *virtual tangent bundle* $_f \in K_0(X)$, which agrees with the class of the cotangent complex of f

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- 3 equivalent formulations:
  - **purity transformation** $f^* \otimes \mathrm{Th}(\tau_f) \to f^!$
  - **fundamental class** $\eta_f \in \mathbb{E}_0(X=Y; \tau_f)$
  - **Gysin morphisms** $\mathbb{E}_n(Y=S; v) \to \mathbb{E}_n(X=S; \tau_f + f^* v)$

  all compatible with compositions

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- Morally, these operations contain the information of \intersecting cycles over X with Y"

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- The construction uses the *deformation to the normal cone*

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Euler class and excess intersection formula

- The **Euler class** of a vector bundle $V \to X$ is the map $e(V) : \mathbb{1}_X \to Th(V)$ induced by the zero section seen as a monomorphism of vector bundles

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where p and q are lci, we have $\quad_p = \quad_q \cdot e(\quad)$, where $\quad$ is the *excess bundle*

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- **Motivic Gauss-Bonnet formula** (Levine, Deglise-J.-Khan) For $p : X \to S$ a smooth and proper morphism

$$(X = S) = p \ e(T_p)$$

where $(X = S)$ is the *categorical Euler characteristic*

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The absolute purity property

- We say that an absolute spectrum $\mathbb{E}$ satisﬁes **absolute purity** if for any closed immersion $i : Z \to X$ between regular schemes, the purity transformation $\mathbb{E}_Z \otimes \mathrm{Th}(_f) \to f^! \mathbb{E}_X$ is an isomorphism

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- Example: the algebraic K-theory spectrum **KGL** satis es absolute purity because K-theory satis es localization property (also called devissage, due to Quillen)

$$K(Z) \to K(X) \to K(X - Z)$$

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- From this property Cisinski-Deglise deduce that the rational motivic Eilenberg-Mac Lane spectrum **H**$\mathbb{Q}$ also satisfies absolute purity, mainly because **H**$\mathbb{Q}$ is a direct summand of **KGL**$_{\mathbb{Q}}$ by the Grothendieck-Riemann-Roch theorem

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The Main result

Theorem (Deglise-Fasel-J.-Khan):
The rational sphere spectrum $\mathbb{1}_{\mathbb{Q}}$ satisfies absolute purity.

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First reductions:

- The "switching factors" endomorphism of $\mathbb{P}^1 \wedge \mathbb{P}^1$ induces a decomposition of the sphere spectrum $\mathbb{1}_{\mathbb{Q}}$ into the direct sum of the plus-part $\mathbb{1}_{+,\mathbb{Q}}$ and the minus-part $\mathbb{1}_{-,\mathbb{Q}}$ (Morel)

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- The +-part $\mathbb{1}_{+,\mathbb{Q}}$ agrees with **H**$\mathbb{Q}$ (Cisinski-Deglise)

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The Main result

Theorem (Deglise-Fasel-J.-Khan):
The rational sphere spectrum $\mathbb{1}_{\mathbb{Q}}$ satis es absolute purity.

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- The +-part $\mathbb{1}_{+,\mathbb{Q}}$ agrees with **H**$\mathbb{Q}$ (Cisinski-Deglise)
- Therefore it su ces to show that the minus part satis es aboslute purity

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The rst proof

- By a devissage theorem of Schlichting and an argument similar to the case of **KGL**, one can show that the Hermitian K-theory spectrum **KQ** satis es aboslute purity

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- This proves the absolute purity of $\mathbb{1}_{\mathbb{Q}}$ when 2 is invertible on the base scheme, since **KQ** is only well-defined in this case

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The second proof

- For every scheme X, denote by $\quad_X : X_{\mathbb{Q}} = X \times_{\mathbb{Z}} \mathbb{Q} \to X$

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The second proof

- For every scheme X, denote by $\varphi_X : X_{\mathbb{Q}} = X \times_{\mathbb{Z}} \mathbb{Q} \to X$
- **Key lemma**: the functor $\varphi_X^* : \mathbf{SH}(X_{\mathbb{Q}})_{\star,\mathbb{Q}} \to \mathbf{SH}(X)_{\star,\mathbb{Q}}$ is an equivalence of categories

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- We may assume that X is the spectrum of a eld, because the family of functors $i_x^!$ for $i_x : \mathrm{Spec}(k(x)) \to X$ for all points x of X is jointly conservative, i.e. re ects isomorphisms

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The second proof

- For every scheme X, denote by $\rho_X : X_{\mathbb{Q}} = X \times_{\mathbb{Z}} \mathbb{Q} \to X$
- **Key lemma**: the functor $\rho_X : \mathbf{SH}(X_{\mathbb{Q}})_{\,,\mathbb{Q}} \to \mathbf{SH}(X)_{\,,\mathbb{Q}}$ is an equivalence of categories
- We may assume that X is the spectrum of a field, because the family of functors $i_x^!$ for $i_x : \mathrm{Spec}(k(x)) \to X$ for all points x of X is jointly conservative, i.e. reflects isomorphisms
- For X a field of characteristic zero, ρ_X is automatically an isomorphism; for X a field of positive characteristic, $\mathbf{SH}(X)_{\,,\mathbb{Q}}$ vanishes by a theorem of Morel/Bachmann

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- For every scheme X, denote by $_X : X_{\mathbb{Q}} = X \times_{\mathbb{Z}} \mathbb{Q} \to X$
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- For X a eld of characteristic zero, $_X$ is automatically an isomorphism; for X a eld of positive characteristic, $\mathbf{SH}(X)_{,\mathbb{Q}}$ vanishes by a theorem of Morel/Bachmann
- The key lemma then reduces the absolute purity of $\mathbb{1}_{,\mathbb{Q}}$ in mixed characteristic to the case of $\mathbb{Q}$-schemes, which can be proved using Popescu's theorem: a closed immersion of a ne regular schemes over a perfect eld is a limit of closed immersions of smooth schemes

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Some applications

- Our method can be used to deduce the following new results in mixed characteristic:
  - The six functors preserve constructible objects in the rational stable motivic homotopy category $\mathbf{SH}(\cdot; \mathbb{Q})$
  - The Grothendieck-Verdier duality holds for $\mathbf{SH}(\cdot; \mathbb{Q})$
  - The homotopy t-structure on $\mathbf{SH}(\cdot; \mathbb{Q})$ behaves as expected

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- Related work: absolute purity of the sphere spectrum over a Dedekind domain (Frankland-Nguyen-Spitzweck, work in progress)

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Thank you!