LONG- E M HI O AND EPHEMERAL CONFIG A ION

CA HE I E G LD EI

Abstract. M fi fi H fi

"Mathematics is the art of giving the same name to different things," wrote Henri Poincaré at the very beginning of the twentieth century (Poincaré, 1908, 31). The sentence, to be found in a chapter entitled "The future of mathematics," seemed particularly relevant around 1900: a structural point of view and a wish to clarify and to firmly found mathematics were then gaining ground and both contributed to shorten chains of argument and gather together under the same word phenomena which had until then been scattered (Corry, 2004). Significantly, Poincaré's examples included uniform convergence and the concept of group.

1. Long-term histories

But the view of mathematics encapsulated by this—that it deals somehow with "sameness"– has also found its way into the history of mathematics. It has been in particular a key feature (though often only implicitly) in the writing of most long-term histories. In the popular genres of the history of π or of the Pythagorean theorem from Antiquity to present times, is hidden the idea that, despite changes in symbolism, the use or not of figures, tables or letters, the presence or not of proofs, some mathematical thing is indeed the same. That, for example, it is thus interesting to extract from behind all its masks the computations of a certain quantity, say the ratio of a circle's circumference to its diameter, even before the quantity may have been baptised π, before it has been described as being a number, or even a well-defined ratio in Euclidean geometry. In more sophisticated versions, by telling the story of a series of past events which have led to finally define an object in our present, history of mathematics may convey the idea that a series of past objects were more or less the same, and have finally been subsumed under the same name. That there is an identity to be detected through or behind contingent disguises concerns not only numbers or simple statements but also whole domains like algebra or statistics, or advanced concepts like group or methods of

---

2010 *Mathematics Subject Classification.* 01A55, 01A85, 11-03, 15-03, 53-03.
Key words and phrases. - , H
I J E .

proof. The present of mathematics speaks for its past on various scales: besides the present formulation of a statement or an object, it also defines the implicit norms for mathematical activities, for instance that they should involve proofs (even if unwritten), or that disciplines are more or less fixed. Debates of course may have be launched and corrections been made about the point of departure, the real origin: did Euclid's *E* [illegible] or Babylonian tablets contain algebra, should we begin only with al-Khwārizmīor François Viète? But there were rarely doubts raised about the relevance of the question itself and the linear character of the history constructed under the key premise of an identifiable thing running all the way long. Many would agree with André Weil that: "More often than not, what makes [history] interesting is precisely the early occurrence of concepts and methods destined to emerge only later into the conscious mind of mathematicians; the historian's task is to disengage them and trace their influence or lack of influence on subsequent developments" (Weil, 1980, 231-232).

There are several good reasons to adopt such a point of view. For one, it is close to the "spontaneous history" of the working mathematician, the chronology often given at the beginning of a mathematical article in order to motivate the results it establishes.[1] The topics of such a history are also more easily those of primary interest to mathematicians (concepts, theorems, in particular). That mathematics deals with long-lived objects may also help to consolidate its status, at times when the importance of mathematics is questioned or the population of students in mathematics is decreasing; that "mathematical truths have been called eternal truths . . . [because] in very different expressions, one can recognise the same truths," in Hieronymus Zeuthen's terms (Lützen and Purkert, 1994, 17), guarantees a particular value for the discipline as a whole. Such a conviction may also reinforce (or be reinforced by) Platonist views of mathematics, and as such has been taken over by some philosophers. For instance, the philosopher of mathematics Jacques-Paul Dubucs, after pointing out the differences between two presentations of a proof that there are infinitely many prime numbers, one in Euclid, one modern by Godfrey H. Hardy and Edward M. Wright, claimed that proper epistemological investigation should focus on what is perceived as a stable and identical core and in particular has "no reason to discriminate the two texts which propose [illegible] the same proof," emphasising his agreement on this issue with the authors of the modern text (Dubucs, 1991, 41).

2. Ephemeral configurations

In 1988, William Aspray and Philip Kitcher noticed that "a new and more sophisticated historiography has arisen [. . .] This historiography measures events of the past against the standards of their time, not against the mathematical practices of today" (Aspray and Kitcher, 1984, 24-25). Indeed, innovative approaches in the history of mathematics of the last decades have often expressed misgivings over a cleaned-up history, based on a too-rapid identification of a concept or a problem, with its historiographical consequences. After giving one possible fifty-year history of the theory of the structure and representation of semi-simple Lie algebras based

[1] [illegible] ([illegible] , 1994).

on four names (Sophie Lie, Wilhelm Killing, Elie Cartan, Hermann Weyl), Thomas Hawkins comments: “[this account] is almost completely devoid of historical content. [. . .] The challenge to the historian is to depict the origins of a mathematical theory so as to capture the diverse ways in which the creation of that theory was a vital part of the mathematics and mathematical perceptions of the era which produced it” (Hawkins, 1987). Consequently, the focus has been much more on localised issues, short-term interests and ephemeral situations, “the era which produced” the mathematics in question ; and moreover it has centred on diversity, differences and changes.

Confluent factors are here at stake. One has been largely advertised. It is linked to contemporary debates in the history of science in the large and comes with the wish to take into account social aspects of mathematics and “how they shape the form and the content of mathematical ideas” (Aspray and Kitcher, 1984, 25), while dimming the line between so-called internal history (that of concepts and results) and external history (that of institutions or scientific politics). It could mean drawing precise connections or parallels between developments in mathematics and in contemporary social or political events. Given the quantity of recent historical writing on these issues, I shall only mention a few examples. The unification of Italian states at the beginning of the second half of the nineteenth century and the cultural *R* which accompanied it favoured a flourishing of mathematics, in particular a strong renewal of interest in geometry in all its forms, with Luigi Cremona, Corrado Segre, Guido Castelnuovo or Eugenio Beltrami and their followers (Bottazzini, 1994; Bottazzini and Nastasi, 2013; Casnati et al., 2016). The Meiji Restoration in Japan witnessed a multifaceted confrontation between the then extremely active, traditional Japanese mathematics ( ) and its Western counterparts (Horiuchi, 1996). The First World War, a “war of guns and mathematics,” as one soldier described it, did not just kill hundreds of mathematicians (among many millions of others) on the battlefields: it also launched entire fields on a vast new scale, such as fluid mechanics or probability theory, and completely reconfigured international mathematical exchanges (for instance fostering a development of set theory, logic and real analysis in newly independent Poland) (Aubin and Goldstein, 2014). One might also think of the variety of national circumstances which preceded the creation of mathematical societies in the late nineteenth and early twentieth centuries (Parshall, 1995) or the various reforms in mathematical education (Karp and Schubring, 2014).

At a smaller scale, specific opportunities at specific times, putting mathematicians in close contact with certain milieux have hosted particular, sometimes unexpected, mathematical work, be they analysis in administrative reforms (Brian, 1994), number theory in the textile industry (Decaillot, 2002) or hardware (De Mol and Bullynck, 2008), or convexity in the military (Kjeldsen, 2002). In such cases (and in many others studied in detail over the last decades), it is not a question of superficial analogies or obvious applications; very often the ways these connections were made, the concrete manner of transmission of knowledge through personal or institutional links, the objectives pursued, are what provides impulse to a mathematical investigation, explains the formulation it takes or the particular balance between computations and theory it displays (Kjeldsen, 2010; Tournès, 2012). This is, by definition, ephemeral in the sense that it implies links with social situations which have their own time scale and are most certainly not “eternal truths.”

However, another reason for this shorter-term focus has recently become even more decisive, to wit, a more acute sensibility to the multi-layered structure of mathematics and the need to study more carefully its variegated components. At first sight, it seems simple enough: mathematics is most often inscribed in texts—although ethnomathematicians also study it directly in strings or sand (Ascher, 1991; d'Ambrosio, 2000)—, it uses words, symbols and drawings, it defines or studies certain objects, it states results and justifies them. It could seem that all we have to do is to decipher the texts and explain the objects, the results and the proofs. Of course, that there were at times debates among mathematicians about certain proofs or objects is well-known: even as late as early modern times, some would not accept a proof based on an algebraically-expressed relation and required geometric proofs in the Euclidean style, as more solid; others dismissed proofs by contradiction or, later, non-constructive proofs; the legitimacy of negative numbers or of functions without derivatives or of sets has been put into question. And around 1900, how to found mathematics—on axiomatics, on integers, or on logic—was a topic of heated controversy among mathematicians, which in turn has been studied by historians.

But more recently, other aspects have been explored, aspects which are not necessarily linked to public and noisy debates, but are part and parcel of ordinary mathematical activities. What is or could be an object or a result, how are they chosen or defined or justified, has changed in time; it has also been seen as depending on the place or the author. What sort of question is considered interesting, by whom and why; which criteria are required to make an argument convincing or a solution satisfactory, again for whom and why; all these aspects and their relations are worth being studied for their own sake. Historians have, for instance, shown that arguments in words or symbols may rely upon, or be inspired by, or sometimes even been replaced by figures, diagrams, tables, instruments.[2] That an acceptable answer to a problem may be, at times, and for certain groups of mathematicians, a single number, an explicit description of all the solutions, an equation, an existence theorem, or the creation of a new concept (Goldstein, 2001; Chorlay, 2010; Ehrhardt, 2012). The way various domains are defined and interact, or are perceived as distinct, has also changed within mathematics, but also between mathematics and other domains, in particular physics (Archibald, 1989; Gray, 1999; Schlote and Schneider, 2011).

What are called "epistemic values," the internalised criteria of what constitutes good mathematics at one time, have also been studied: rigour is the most obvious perhaps and has a complex history, but effectiveness or generality or naturalness may appear at some moments to be even more decisive (Mehrtens, 1990; Rowe, 1992; Schubring, 2005; Corry, 2004; Chemla et al., 2016). Working mathematicians have usually their own answer to these questions, but the point here is to reconstruct the whole range of positions at a given time, in a given milieu, and to understand their effect in mathematical work. For instance, in the dispute between Leopold Kronecker and Camille Jordan in 1874 about what we now see as the [illegible] reduction theorem for matrices, differences in formulation (elementary divisors on one side,

[2] F[illegible] ff[illegible] ([illegible], 1999; L[illegible] 2014; M[illegible], 2012; M[illegible], 2003; D[illegible]-[illegible], 2010; [illegible], 2011).

canonical form on the other), and in disciplines (invariant theory vs group theory) were at play, but also in conceptions of generality (Brechenmacher, 2016).

Last, but not least, the way mathematics is made public and circulates has been proved to be both a serious constraint on its form and an essential factor in its transmission; the organization of correspondence among mathematicians (indeed the mere form of a mathematical letter), the creation of mathematical research journals in the nineteenth century and their different organisation through time, the advent of academies, seminars and conferences, teaching programs and textbooks, for instance, have all been scrutinised (Ausejo and Hormigón, 1993; Peiffer, 1998; Schubring, 1985a; 1985b; Verdier, 2009; Gérini, 2002; Remmert and Schneider, 2010). The last two aspects could also be considered as links between mathematical developments and general cultural issues, but here the emphasis is on the combination of these components inside mathematical texts themselves.

Depending on one's own tastes, the sheer variety of them in the course of time may appear fascinating or an irrelevant antiquarian interest. However, we now have enough evidence that all these aspects may count for understanding the development of mathematics. Mathematics weaves together objects, techniques, signs of various kinds, justifications, professional lifestyles, epistemic ideas, and these components have distinct time-scales. Changes in the various components do not occur simultaneously. One can of course, study these aspects separately, as witnessed by several recent projects; often in a comparative way, and to display their variety. But even when one is able to understand a long-term development of one component (for instance, of mathematical publishing), its articulation with other components is generally stable only over a shorter period—recent biographies, indeed, offer successful examples of the study of such articulations, see (Parshall, 2006; Crilly, 2006; Alfonsi, 2011).

A further difficulty is that if concepts or theorems are aspects of which the mathematician is aware (and very much so), some of the components I mentioned are much more implicit, or are operational at a collective, not at an individual, level and can be best detected and analysed for an entire group (Goldstein, 1999). All this explains the interest in studying what I am calling here "configurations," a word borrowed from the sociologist Norbert Elias. Elias wanted to set himself apart from previous sociological theories based on an a priori hierarchic opposition between individuals and society, and he promoted the idea of first studying configurations formed by interactions between persons in interdependence, at different scales, be they players in a game or workers in an enterprise. For us, configurations are between texts or persons, coordinating some of the components we have mentioned (we shall see other examples later).

A last reason has favoured more localised studies: a critical outlook by historians of mathematics on their own practice. For a long time, words like "discipline" or "school" have been used without further ado, in particular because they were terms inherited from mathematicians of the past themselves. Recent work has shown that to define and use them more carefully gave a better grasp for describing the past. For instance, using a characterisation of a discipline as a list of internal elements (core concepts, proof system, etc) provided by Ralf Haubrich, Norbert Schappacher and I were able to distinguish in the lineage of C. F. Gauss's *D* -
A those parts which fused into a existing discipline (his treatment of the cyclotomic equation, for instance, which had a potent effect on the theory of

equations), those parts which emancipated themselves as autonomous disciplines with their own programmes and priorities (quadratic forms in the middle of the nineteenth century, reciprocity laws in the theory of number fields later), and those parts which, in the nineteenth century, did not (primality tests), even if they were taken over in an isolated manner by some mathematicians.[3] Caution also applies to common descriptors of historical phenomena themselves, such as "context" (Ritter, 2004), " " (Aubin and Dahan Dalmedico, 2002) or "revolution" (Gillies, 1992). This reflexivity has also permitted historians to find counter-examples to overly-crude hypotheses on the long-term development of mathematics (Gilain and Guilbaud, 2015).

Let us now return to long-term histories. It may happen that the statement or the object we are interested in bears obvious traces of its time. For instance, let us consider the following three sentences : "the area of a right-angled triangle in integers cannot be a square," "the equation $\;^4 - ,^4 = \;^2$ has no non-trivial rational solutions," "the plane projective curve defined over $\mathbb{Q}$ by $,^2 = \;^3 - \;^2$ has exactly four points with rational coordinates." The first is the form under which it was proved by Pierre Fermat around 1640; the second is the form under which the first was often presented after the eighteenth century; the third is a special case of Louis Mordell's 1922 theorem on rational points of elliptic curves. Different geometrical objects, even different types of geometry, and different equations are involved. But as explained by André Weil, not only the statements, but the proofs themselves, can be identified and seen as , all expressing the fact that the Mordell-Weil group of the elliptic curve defined by $,^2 = \;^3 - \;^2$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Such retrospective identification, then, can be used as a basis for tracing a long-term history of any of these three statements; or, more in line with current historiography, it can be seen as a historical problem , which requires first the reconstruction of various configurations involving each of the statements and, if possible, how they have come to be identified.[4]

Another case is even more delicate and can be illustrated by the theorem that there are exactly twenty-seven lines on a non-singular cubic projective surface; since its statement (and proof) in 1849 by both Arthur Cayley and George Salmon, its formulation has remained remarkably stable for more than a century. But what changed is its association with other problems: as shown in (Lê, 2015), it is for instance in tandem with the fact that there are 9 inflection points on a cubic projective plane curve and other analogous statements that it played a decisive role for the assimilation of group-theoretical methods by geometers Felix Klein's Erlangen Program; this specific configuration of questions and disciplinary issues, around the so-called "equations of geometry," lasted only some years, but was a key feature in the transmission of the theorem.

Concepts or theories, instead of statements, may also come with a diversity of formulations: we may think of ideals, points or Galois theory (Edwards, 1980; 1992; Schappacher, 2010; Ehrhardt, 2012). But I would like to discuss in some detail the

---

[3](G , 2007), (G , 2009; 2010). "," (, 2003).

[4] (G , 1995). F (G , 1991; , 1994; B , 2007).

concrete case of a rather stable concept, that of an Hermitian form.[5] From all accounts, for instance (Vahlen, 1900, 612-613) or (Dickson, 1919, vol. 3, p. 269), forms of this type first publicly appear in 1854, in an article authored by Charles Hermite (Hermite, 1854) and we shall begin with it.

3. Hermite's version of Hermitian forms

This paper, finished in August 1853, is the last of three memoirs devoted to forms published by Hermite in the same issue of *C ' J* . At the very beginning, Hermite explains (Hermite, 1854, 343):

> One knows how easily one can extend the most fundamental arithmetical concepts coming from to complex numbers of the type $a + b\sqrt{-1}$. Thus, starting from elementary propositions concerning divisibility, one quickly reaches those deeper and more hidden properties which rely upon the consideration of quadratic forms, without changing anything essential in the principle of methods which are proper to real numbers. In certain circumstances, however, this extension seems to require new principles and one is led to follow in several different directions the analogies between the two orders of arithmetical considerations. We would like to offer an example to which we have been led while studying the representation of a number as a .

No references are provided in this introduction, but the background seems rather clear, both then and now. Carl Friedrich Gauss, searching for an extension to higher powers of the reciprocity law for squares that he had proved in his 1801 *D A* , launched, in 1831, an arithmetical study of what he called "complex integers," (now Gaussian integers), that is "complex numbers of the type $a + b\sqrt{-1}$" with a and b ordinary integers (Gauss, 1831/1863). Among them he defined prime complex numbers, units $(\pm 1, \pm\sqrt{-1})$, proved the factorisation of the "complex integers" into a product of these prime numbers (unique up to units and the order of the factors), showed the existence of a Euclidean division and extended congruences to these numbers: in short, "the elementary propositions which concern divisibility." In 1842, Peter-Gustav Lejeune-Dirichlet began to study of "these deeper and more hidden properties which rely upon the consideration of quadratic forms," in particular the representation of "complex integers" by what will come to be known as Dirichlet forms at the end of the century, that is, quadratic forms $f(\ , ,) = a \ ^2 + 2b \ , + c, ^2$, where the coefficients a, b, c, and eventually the values taken by the indeterminates and , , are also "complex integers" (Dirichlet, 1842).

However, such a discontinuous chronology (1801–1831–1842–1853) is not sufficient to understand the background of Hermite's work. Gauss's discussion of his complex integers appeared in latin in the proceedings of the Göttingen Society of Science. As early as 1832, Dirichlet explained Gauss's work and completed it in what was at the time the only important journal entirely devoted to mathematics, August Leopold Crelle's *J M* , created in 1826. Dirichlet, who had spent several years in Paris, was an important

[5] I K fi $^{-}($, , $K = \mathbb{C}$), H f K- V $f(u, v) = f(\bar{v}, u)$, u v V.

go-between for mathematics: his 1832 article, written in French, was clearly aimed at an international audience. In the same decade, he would use new tools developed by analysts, in particular Fourier series, to complete proofs of Gauss's statements and a number of his arithmetical results would be published both in German and in French, in a way that would draw greater attention to them. In 1840, a letter to Joseph Liouville, on the occasion of a French translation of one of his papers, already announced to the French community his current interest for "extending to quadratic forms with complex coefficients and indeterminates, that is, of the form $+ \sqrt{-1}$, the theorems which occur in the ordinary case of real integers. If one tries in particular to obtain the number of different quadratic forms which exist in this case for a given determinant, one arrives at this remarkable result, that the number in question depends on the division of the lemniscate; exactly as in the case of real forms with positive determinant, it is linked to the division of the circle" (Dirichlet, 1840). The lemniscate pointed to the integral $\int \frac{dx}{\sqrt{1-x^4}}$ and to elliptic functions, then at the forefront of research. Dirichlet's letter was even reproduced twice: in Liouville's *J* (created in 1836) and in the *C* (created in 1835).

The ten years preceding Hermite's 1853 paper were indeed turbulent years for complex functions and numbers, and for quadratic forms. Also with reciprocity laws in view, several authors tried to tackle numbers of the type $+ \sqrt{a}$, for and integers and a an integer without square divisors. In a footnote of his 1832 paper, Dirichlet announced, somehow optimistically, that they give rise to theorems analogous to those on Gaussian integers, and with similar proofs. In 1839 (with a French translation in Liouville's journal three years later), Carl Jacob Jacobi, again recalling Gauss's theory of complex integers, showed that a prime number $= 8n + 1$ can be written not only as a product of two prime Gaussian complex integers $(a+b\sqrt{-1})(a-b\sqrt{-1})$ and thus represented by the quadratic form a^2+b^2, but also as $(c+d\sqrt{-2})(c-d\sqrt{-2}) = c^2+2d^2$, as well as $(e+f\sqrt{2})(e-f\sqrt{2}) = e^2-2f^2$. Through computations, Jacobi rewrote as a product of complex numbers, each of them a linear combination with integral coefficients of powers of a given 8^{th}-root of unity: the three combinations of these four numbers, two by two, provide the three decompositions of noted by Jacobi, but this time, as he writes, "from a common source." Announcing similar results for a prime $= 5n+1$ (and 5^{th}-roots of unity), but with no hint of a proof, Jacobi provided the spur for decisive work by several younger mathematicians in the 1840s. These included Ernst Eduard Kummer's theory of ideal numbers (in what we call now cyclotomic rings), Gotthold Eisenstein's approach to complex multiplication, and Hermite's own first research on quadratic forms, concerning which he wrote directly to Jacobi from 1847 on.[6] Personal relations here reinforced the circulation of the articles; Hermite was informed of Kummer's approach to the arithmetic of complex numbers by the mathematician Carl Wilhelm Borchardt during the latter's 1847 Parisian tour and he met Dirichlet and Eisenstein, among others, during his own trip to Berlin at the beginning of the 1850s.

Kummer's display that unique factorization failed in general certainly crushed Dirichlet's 1832 hopes and showed that "In certain circumstances, this extension [of arithmetic to complex numbers] seems to require new principles." Hermite's

6 (G , 2007, 39-51).

forms arose at least as much from his close reading of Jacobi as from Dirichlet's work on complex numbers. In letters to Jacobi on quadratic forms, published in 1850, Hermite had considered forms with any number of indeterminates and real coefficients (instead of the two indeterminates and integral coefficients of Gauss's *D*). His main result was to establish that there exists a (non-zero) value of the form, when evaluated on integers, which is less than a certain bound, depending on the number of indeterminates and on the determinant of the form, but not of its coefficients.[7] In particular:

H ' M

Let $f({}_0, {}_1 \cdots, {}_n)$ a definite positive quadratic form with $n+1$ indeterminates and real coefficients, there exist $n+1$ integers α, β, $\cdots$, λ

$f(\ , \ ,)$ is not 0 and is less that $1, 15$; thus it should be exactly . And thus the number $= (\quad - \alpha,)^2 + , ^2$ is a sum of two squares. Hermite also adapted these forms to discuss the divisors of forms of the type $^2 + A, ^2$, which also played a role in Jacobi's 1839 paper, and it seemed a natural step to adapt them also for the famous theorem that every integer is the sum of four squares.

Let A be a non-zero integer and let assume without loss of generality that 4 does not divide A. Hermite first showed that one can find integers α and β such that $\alpha^2 + \beta^2 \equiv -1 \bmod (A)$.

He then introduced the quadratic form with 4 variables:

$$f(\ , \ , \ , \ ) = (A \quad + \alpha \quad + \beta \ )^2 + (A, \ - \beta \quad + \alpha \ )^2 + \ ^2 + \ ^2$$

other, those which can be expressed as:

$$
\begin{aligned}
&= aV + bW \\
{}_0 &= a_0V_0 + b_0W_0 \\
&= cV + dW \\
{}_0 &= c_0V_0 + d_0W_0
\end{aligned}
$$

> where a, b, c, d are arbitrary imaginary numbers and a_0, b_0, c_0, d_0 their conjugates. Thus one obtains a perfectly defined class of real transformations.

In a modernised matrix notation, these transformations are such that

$$\begin{pmatrix} \\ \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}$$

for a, b, c, d complex numbers, which is the same as a specific transformation with 4 real variables:

$$\begin{pmatrix} \\ , \\ \\ \\ \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\,(a) & -\mathrm{Im}\,(a) & \mathrm{Re}\,(b) & -\mathrm{Im}\,(b) \\ \mathrm{Im}\,(a) & \mathrm{Re}\,(a) & \mathrm{Im}\,(b) & \mathrm{Re}\,(b) \\ \mathrm{Re}\,(c) & -\mathrm{Im}\,(c) & \mathrm{Re}\,(d) & -\mathrm{Im}\,(d) \\ \mathrm{Im}\,(c) & \mathrm{Re}\,(c) & \mathrm{Im}\,(d) & \mathrm{Re}\,(d) \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ U \end{pmatrix}.$$

It is at this point that Hermite introduced what we now call Hermitian forms, for him forms of the type:

$$f(\, , \,) = A \quad {}_0 + B \quad {}_0 + B_0 \quad {}_0 + C \quad {}_0 \tag{3}$$

where A et C are real numbers, and B a complex number (B_0 its complex conjugate).

If one replaces the complex variables and by their real and imaginary parts, that is by real variables, one finds:

$$f(\, , , , \, , \,) = A(\, ^2 + , ^2) + 2\,\mathrm{Re}(B)(\quad + , \,) + 2\,\mathrm{Im}(B)(\quad - \, ,) + C(\, ^2 + \, ^2),$$

of which the form (2) used to prove the theorem of four squares is a prototype.

> Considered with respect to the original variables , , , , , these forms are entirely real, but their study, with respect to the transformations we have defined previously, essentially relies upon the use of complex numbers. One is then led to attribute to them a mode of existence singularly analogous to that of binary quadratic forms, although they essentially contain four indeterminates (Hermite, 1854, 346).

If the linear transformations he had "defined previously" have a determinant of complex norm 1, they leave invariant the quantity $\Delta = BB_0 - AC$, which, thus, will play the role of the determinant for the form (3). Hermite then proceeds to classify the forms with a fixed determinant up to the special linear transformations he has selected, provides a reduction theory for them, uses them to approximate complex numbers by quotients of Gaussian integers, etc.

I would like to underline two important features. The first one is the central role played by the linear transformations: this is the restriction on these transformations that defines the new type of forms. In several memoirs around 1850, Hermite and some of his contemporaries, Arthur Cayley, Borchardt, Eisenstein, for instance, defined several types of equivalence among forms, depending on the transformations which are taken into account: either they look for transformations which keep a given form or function invariant, or, fixing a group of transformations, they study the forms which are left invariant by them. In 1855, in a study on the transformations of Abelian functions with 2 variables (more precisely of their periods), Hermite again introduces other "particular forms with 4 indeterminates, where one does not use as analytical tool the most general transformations among 4 variables, but particular transformations ... which reproduce analogous forms" (Hermite, 1855b, 785).

This point of view, changing the group of transformations operating on the variables into forms or functions to delineate which type of forms will be studied, was at the time tightly linked to invariant theory (Parshall, 1989). Its applications, again, are varied, from the law of inertia for quadratic forms to Sturm's theorem on the number of roots of an equation that belong to a certain domain. It is in this context that Hermite generalised his 1853 construction for $n = 2$ to quadratic forms with $2n$ "pairwise conjugate" indeterminates,

$$f(\ _1,\ _2,\dots,\ _n,\ _{1,0},\dots,\ _{n,0}) = \sum_{i,j} a_{i,j}\ _i\ _{j,0},$$

with $a_{i,j}$ and $a_{j,j}$ complex conjugates (thus $a_{i,i}$ real numbers) (Hermite, 1855a; 1856) ; he called them "quadratic forms with conjugate imaginary indeterminates."

The second key point is about the introduction of new objects. Hermite's "forms with conjugate imaginary indeterminates" quadratic forms of a specific type, not a new type of objects defined in an ad hoc way to accommodate complex numbers. They are distinguished a larger class of well-known objects because of their special properties (here their stability under a certain group of transformations). For Hermite, as well as many other mathematicians of the nineteenth century, mathematicians do not, should not, create their objects: they "meet them or discover them and study them, like physicists, chemists and zoologists" (Hermite and Stieltjes, 1905, vol. 2, p. 398). This conviction accompanies the emphasis on classification, which in turn permits an assimilation of mathematics to a natural science (Lê and Paumier, 2016). "Collecting and classifying" was also a very strong incentive for invariant theorists like Cayley (Crilly, 2006, 193-195), but not limited to them, nor to the 1850s: in 1876, still, Leo Königsberger wrote for instance: "It seems to me that the main task now just as for descriptive natural history consists in gathering as much material as possible and in discovering principle by classifying and describing this material" (File H1850(6), Staatsbibliothek zu Berlin, Handschriftenabteilung).

Thus the configuration about the appearance of Hermitian forms I have just briefly sketched include local incentives, a series of specific themes, a collection of objects and the disciplinary tools available to study them (here for instance the reduction theory of forms), the state of the art on certain topics (for instance complex numbers) and general points of view on mathematics. Some of these components will evolve separately and at different rates (an obvious instance is the decomposition of numbers into irreducible complex factors), losing their connections with the development of our forms "with conjugate imaginary indeterminates."

Tracing, on the other hand, the fate of this particular type of forms until the 1920s, where they made a spectacular appearance in quantum theory (Born and Jordan, 1925) is not obvious. In 1866, for instance, Cayley discusses "Hermitian matrices" and associated forms, but they are those attached to the transformation of Abelian functions, not "our" forms and transformations(Cayley, 1866). Several "Hermitian forms" thus appear ephemerally during the century. The variety of notations for complex conjugation also forbids to look for a visual display.

A technique which has recently demonstrated its effectiveness is to systematise the search in review journals. For the second half of the century, the *J* *F* *M* which began in 1868 can be used, but it did not offer "Hermitian form" as a heading; looking for "Hermitian" in the Basic Index of its digitalised version is misleading as it includes many articles reindexed in the current MSC classification in 11-E39 (Bilinear and Hermitian forms) or 11E41 (Class numbers of quadratic and Hermitian forms) but with no relation to Hermitian. I thus had recourse to several other databases and also to Thamous, an electronic medium for a relational database developed by Alain Herreman, which allows the registration of links between articles (for instance, but not only, through cross-references). For reasons of space, and with a view to the long-term question, I shall only briefly report on some of the topics in which forms "with conjugate imaginary indeterminates" occur before the First World War.

4. Picard's and Bianchi's groups

The first topic, chronologically, involves Émile Picard and Luigi Bianchi, both born in 1856. Picard is best known for his work on algebraic surfaces (Houzel, 1991, 245) and, more generally, for extending to dimension 2 a number of results first established in dimension 1. In 1881 for instance, he proved, in parallel with the 1-dimensional case of the (elliptic) curve, that surfaces which can be parametrised by Abelian functions have (under some restrictions on their singularities) a geometric genus $_g \leq 1$ (Picard, 1881).

But at the very same time and in close proximity, a great enterprise was underway; Henri Poincaré had just developed his theory of "Fuchsian" and "Kleinian" (now both considered particular cases of "automorphic") functions, that is, functions of one complex variable invariant under certain groups of transformations. More specifically, one considers a meromorphic function of one complex variable inside a certain disk such that

$$f(\frac{a\ \ +b}{c\ \ +d}) = f(\ ), \tag{4}$$

where the Moebius transformations $\rightarrow \frac{az+b}{cz+d}$ belong to a discontinuous group of invertible transformations; the groups and the functions are "Fuchsian" for Poincaré when the coefficients a, b, c, d are real. An important particular case is that of the modular functions, where the coefficients a, b, c, d are integers, and $ad - bc = 1$. Poincaré had interpreted the Moebius transformations on the unit disk (or equivalently on the upper half plane) as isometries in a non-Euclidean (hyperbolic) geometry and he had constructed fundamental domains for the Fuchsian groups. He had also shown, given a Fuchsian group, how to construct Fuchsian functions invariant under the group and how to connect them to the solution of differential equations (Gray, 2000).

Picard was at first in search of a single example with two variables, one that would be analogous to the modular functions. For this he studied the curve of equation $^3 = (- 1)(-)(- ,)$ and the periods of $= \int {}^{-1} d$ (Picard, 1882a). These periods, as functions of and , , satisfy a system of partial differential equations, which admits a basis of three independent solutions A_1, A_2, A_3. Picard defined the functions , of the two variables and , by $= \frac{A_2}{A_1}$, $= \frac{A_3}{A_1}$, and by inversion, obtained what he was looking for, two uniform functions of the two variables , , defined on the domain $2\,\mathrm{Re}( ) + \mathrm{Re}\ {}^2 + \mathrm{Im}( )^2 < 0$)y)Ae2;alt 2]TJ/9.8'd defined the. vl(as-369401(trans hyperfuchsian, associated to the tesselation of the hypersurface. In later papers, he constructed defined in the interior of the hypersurface and invariant by such a group, showing that hyperfuchsian functions corresponding to the same group can be expressed as rational functions of three of them (linked by an algebraic relation). Picard also developed an analogous arithmetical study of forms with conjugate indeterminates, interpreting their reduction geometrically in terms of domains on the plane limited by arcs of circles and, this way, constructed afresh Fuchsian groups (Picard, 1884).

When he picked up the topic in 1890, Luigi Bianchi was coming from a quite different background; he had just studied the arithmetic of Dirichlet forms with Gaussian-integer coefficients, with the intent to complete Dirichlet's results on the number of classes of such forms. He then proceeded to complete some points in Picard's study of the arithmetic of Hermitian forms, launching both an extension to forms with coefficients in any quadratic field and a detailed examination of the associated groups of transformations and their subgroups of finite index (Brigaglia, 2007). All along, and in particular in his synthesis published in *M* *A* in German, Bianchi handled side by side the arithmetic of Dirichlet and of Hermitian forms and their geometrical interpretations, in particular the determination of their fundamental polyhedra. Bianchi had studied with Felix Klein in Göttingen during his European post-doctoral tour and his main reference in his papers on the arithmetic of forms is Klein's - *M* , completed by Robert Fricke, which had just appeared in 1890. As Bianchi explained it (Bianchi, 1891, 313):

> The geometrical method, on which Professor Klein bases the arithmetical theory of the ordinary binary quadratic forms may be applied with the same success on a larger scale. To prove this is the aim of the following development which will treat in the same way the theory of Dirichlet forms with integral complex coefficients and indeterminates and of Hermitian forms with integral complex coefficients and conjugate indeterminates.

This publication pushed Picard to write to Klein on May 15, explaining how to extract from his own articles some of the results obtained by Bianchi in the case of Hermitian forms with Gaussian integers as coefficients. . . However, a year later, Bianchi would handle forms whose coefficients are integers in quadratic fields $\mathbb{Q}\sqrt{-D}$ for $D = 1, 2, 3, 5, 6, 7, 10, 11, 13, 15, 19$, displaying a good knowledge of Dedekind's theory of ideals. In this paper, he extended the main group of transformations to include those whose determinant is any unity and, following an idea of Fricke, those of the type $\rightarrow \frac{az_0+b}{cz_0+d}$ (the index, as before, indicating the complex conjugation) and he computed the corresponding fundamental polyhedra. Klein and Fricke integrated all these results, mostly from Bianchi's point of view, in their 1897 *F* ; Bianchi's and the number-theoretical part of Picard's results would then be taken up and developed through different methods by a number of mathematicians in the following decades, Onorato Nicoletti, Otto Bohler, Leonard Dickson, Georges Humbert, Gaston Julia, Hel Braun, Hua Luogeng, and many others. The "Bianchi groups" would then be our groups $PSL_2(\mathcal{O}_d)$, for $\mathcal{O}_d$ the ring of integers of the imaginary quadratic field $\mathbb{Q}\sqrt{-d}$, while the "Picard group" would designate the case $d = 1$, that of the Gaussian integers.

5. A Theorem and three authors

Meanwhile, the question of the *fi* subgroups of the linear groups returned to the forefront as the theorem:

For any finite group of n-ary linear homogeneous transformations, there exists an n-ary positive definite Hermitian form, say, $\sum a_{ik}\ _i\ \bar{}_k$, that the group leaves absolutely invariant.[9]

For once, the date of this theorem seems rather precise: July 1896! The precise day, on the other hand, and the author, are another story. On Monday July 20, 1896, it appeared in a note written by Alfred Loewy and presented to the French Academy of Sciences by Picard for insertion in the *C* (Loewy, 1896). Following the tradition of this journal to accept only very short communications, it consisted mainly of an announcement and contained no proofs. Two years earlier, Loewy (1873–1935) had obtained his thesis under Ferdinand Lindemann, with a work on the transformation of a quadratic form into itself. In his 1896 note, Loewy first gives a condition for a linear transformation to fix an arbitrary bilinear form with conjugate indeterminates, $\sum a_{ik}\ _i\ ^0_k$, with arbitrary coefficients (the complex conjugation is here indicated by the exponent "0"). When restricted to a "quadratic form of M. Hermite," that is when $a_{ik} = a^0_{ki}$, it says that the form can be transformed into itself only by transformations whose characteristic polynomials have roots of modulus 1 and simple elementary divisors. Loewy then states the above theorem, and used it in particular to complete a previous study by Picard of the finite groups of ternary linear transformations (Picard, 1887). For each group, except one, Picard had found a quadratic form left invariant by the group, and Loewy provides an explicit invariant Hermitian form in the remaining case. Loewy's detailed work was included in his 1898 *H* at the Albert-Ludwigs-Universität in Freiburg and published in the collection of the Nova Leopoldina, with a summary in the *M A* to which I shall come back (Loewy, 1898). Meanwhile, following immediately after the publication of his note, there was unleashed a flood of publications.

On August 9, Fuchs communicated to the French Academy a note pointing out that what Loewy had published " " (Fuchs's emphasis) was a special case of his own results presented at the Berlin Academy exactly one month earlier, on July 9. Now Fuchs had, for quite some time, been studying differential equations of the type:

$$\frac{d^n\omega}{d\ ^n} + \ _1\frac{d^{n-1}\omega}{d\ ^{n-1}} + \cdots + \ _n\omega = 0$$

where the coefficients $\ _i$ are uniform functions of the variable , with a finite number of poles. Choosing a set of n linearly independent solutions in the neighbourhood of a singularity, the new solutions one obtained when the variable describes a circuit around the singularity can be expressed as linear combinations of the original ones, and Fuchs, among others, had studied these monodromy transformations, in

[9] [illegible] x_i, [illegible] $\bar{x}_i$. A [illegible] $GL_n(\mathbb{C})$ [illegible].

particular their "fundamental equations" (for us, the characteristic equations of the matrices associated to the group of monodromy transformations) (Gray, 1984). In his July presentation, Fuchs stated that, under several assumptions on the roots of these fundamental equations (in particular, that, for one equation at least, the roots are all distinct and of modulus 1), there exists a linear combination with determined real coefficients A_i

$$\phi = A_1\omega_1\omega_1' + A_2\omega_2\omega_2' + \cdots A_n\omega_n\omega_n'$$

of a fundamental system of solutions ω_i of the differential equation (ω_i' being here the conjugate function ...) which is unaltered by the group of monodromy. For algebraically integrable differential equations, the group is finite and Fuchs (like Loewy will do) uses his theorem to complete Picard's work on ternary forms. In his Berlin Academy presentation, even for the finite case, he had explicitly (and quite unnecessarily) assumed that the roots of at least one fundamental equation should be distinct, an assumption which is not repeated in his August note to the French Academy.

From September 21 to September 26, the annual meeting of the Deutsche Mathematiker-Vereinigung took place in Frankfurt and Klein presented a one-page paper (Klein, 1896) which added another author to the theorem. First of all, Klein recalled his own 1875 work (Klein, 1875) where he had explained how to interpret a finite group of complex linear transformations on two variables as a group of real quaternary collineations of the ellipsoid, ${}^2 + ,{}^2 + {}^2 - {}^2 = 0$; and he had shown that this group necessarily fixes a point within the ellipsoid, thus providing a finite group of (real) rotations around a fixed point. This was the basis of his classification in the binary case. The ternary case he attributed not only to Picard, but also to Hermann Valentiner. Valentiner, who after a thesis on space curves had gone to work for a Danish insurance company, while still contributing to mathematics, had indeed published in 1889 a book on the classification of finite binary and ternary groups of linear transformations (Valentiner, 1889). Then, after a nod to Loewy's note, Klein devoted the remainder of his presentation to another, simpler, proof that Eliakim Hastings Moore has communicated to him. The proof is indeed simple. For any Hermitian definite form, the sum of its transformations by the (finitely) many elements of the group is still a Hermitian definite form and it is fixed by the group. It is remarkable enough that Klein had to explain in detail that such a procedure would not necessarily work if the group was infinite... In the written version, Klein adds that Moore had indeed spoken about his theorem at a mathematical meeting in Chicago on July 10 (with a written version published locally on July 24!).[10] He also alluded to Fuchs's work without more details (given the past tensions between Fuchs and Klein, one may think that this vague recognition was not completely satisfactory to Fuchs (Gray, 1984)).

Both Moore and Loewy published an extended version of their respective work in the same issue of the *M* *A*, in 1898. Both men analysed the literature, in particular the various claims made about their theorem. They underlined Fuchs's superfluous condition to refute his claim to the theorem (Loewy emphasising that, above all, the definite character of the invariant form was never

[10] A M C , (, 1994, 399) C .

Hermitian forms. Fubini considered an Hermitian form algebraically equivalent to ${}_1\,{}^0_1 + {}_2\,{}^0_2 + \cdots {}_{n-1}\,{}^0_{n-1} - {}_n\,{}^0_n$ (here again the exponent 0 indicates the complex conjugation) and the associated group transforming a "hypervariety" of the type $\sum_1^{n-1}\ {}_i\,{}^0_i - 1 = 0$ into itself. For two points, and $^-$ (the bar does designate the complex conjugation), he introduced the quantity

$$(6) \qquad R_{u\bar{u}} = \frac{(\sum_1^{n-1}\ {}_i\,{}^{-0}_1 - 1)(\sum_1^{n-1}\ {}^{0}_i\,{}^{-}{}_1 - 1)}{(\sum_1^{n-1}\ {}_i\,{}^{0}_1 - 1)(\sum_1^{n-1}\ {}^{-0}_i\,{}^{-}{}_1 - 1)} - 1.$$

This is invariant under the group of transformations, real and equal to 0 within the hypervariety only when the two points coincide; it is thus legitimate to call it a (pseudo)-distance between the two points. Fubini used it primarily with arithmetical and analytical applications in view (Fubini, 1903), but he also refined his construction in order to interpret the Hermitian form as a metric for a complex space (Fubini, 1904).

One year later, but independently, Study became interested in the same question and in an article published in *M A* (Study, 1905), he also defined Hermitian metrics and distances. His project however was different and, as far as geometry was concerned, more extensive. Study had worked before on invariant theory and quaternions and had just published in 1903 a book on *G D*, using biquaternions and geometrical tools to study mechanical forces.[12] At the beginning of his 1905 paper, he remarks that, while integration and other questions depending on equalities had been successfully extended to the complex realm, this was still not the case for problems of extrema.

His idea then is to develop this study within the framework defined by Corrado Segre who, in 1890, had opened new vistas in complex geometry. Segre had adapted Karl von Staudt's approach to the complex case (Segre, 1890a; 1890b). In particular, while polarities were associated, in the real case, to symmetric bilinear forms, Segre had associated to his newly-found antipolarities, which occur only in the complex case, the forms which, in Segre's words "have also already been introduced in number theory thanks to M. Hermite, M. Picard and others," that is Hermitian forms (Brigaglia, 2016, 275). From a ternary indefinite Hermitian form, $(\ ^-) = {}_1\,{}^-{}_1 - {}_2\,{}^-{}_2 - {}_3\,{}^-{}_3$ for instance (here the bar does designate the complex conjugation), Study, following Segre's concepts if not his terminology, defined a "Hermitian point-complex" by $(\ ^-) = 0$ (Segre's) and the inside of the point-complex by the condition $(\ ^-) > 0$. He was then able to define a hyperbolic Hermitian metric and the (real) distance of two points inside the point-complex. Under an adequate normalization, the distance between two points and , is

$$(\ ,\ ,\ ) = 2\cosh^{-1}\frac{\sqrt{(\ ,^-)(^-,\ )}}{\sqrt{(\ ^-)}\sqrt{(\ ,\ ,^-)}},$$

Study showing then that the distance between two points is the length of the geodesics linking them. He also developed the case of an elliptic Hermitian metric, based this time on a definite Hermitian form. This setting would then be developed by Julian Coolidge, Wilhelm Blaschke and of course Erich Kähler, Jan Schouten and Élie Cartan in the 1920s and 1930s.

12 , , (H, 2005).

7. Back to long-term histories

Configurations based on conflicts of priority are, at least in our case, particularly ephemeral, if not at a personal level, then at least collectively. Issues are very soon settled in the literature; there will be a "Fubini-Study" metric, the theorem on the invariance of a Hermitian form by a finite group of linear transformations will be attributed to all the authors we have discussed above.[13] Yet studying such priority configurations remains of interest, for they are often the easiest kind to spot and they can help us reveal larger or more decisive configurations: such as, for instance, the intense, international work set in motion to describe the finite subgroups of $GL_n(\mathbb{C})$, or the problems raised by the extension of geometry to the complex realm, or the role of Klein as a go-between between Göttingen, Italy and with conjugate indeterminates (or variables)", "forms of M. Hermite," "Hermitian forms," while this last designation is applied also to other types of forms or matrices.

This hint is confirmed by the fact that we do not find them in textbooks; neither in Richard's Baltzer's classic *D* , nor in Heinrich Weber's *L* *A* , nor even in Max Bôcher's 1907 *I* *H* *A*- , which last is devoted to matrices and bilinear and quadratic forms. The situation changes only after the first decade of the twentieth century, with, for instance, the sections on Hermitian forms in Gerhard Kowalewski's *E* *D* in 1909, David Hilbert's 1912 *G* *I* , or Harold Hilton's *H* *L* - of 1914.

Moreover, in the configurations we have sketched, the references prior to 1880 cited by our authors do not mention Hermitian forms (except for a possible, isolated, reference to Hermite himself). Typically, for instance, Picard quotes a 1880 memoir by Jordan on the equivalence and the reduction of Hermitian forms, but there are no Hermitian forms in the articles cited by Jordan himself. In the case of finite groups of transformations, neither Klein in 1875, for the binary case, nor Valentiner in 1889, for the ternary case, uses Hermitian forms (but in the reedition of the former in his , Klein will add a note on the interpretation of his results by Hermitian forms). A rare exception seems an article by Elwin Bruno Christoffel in 1864 (cited by Frobenius in 1883, in an article cited by Loewy) (Christoffel, 1864): Christoffel considers bilinear forms $\phi = \sum [gh]\ _g\ _h$ and their values when the values of $_g$ and $_g$ are complex conjugate numbers, under the assumption that the coefficients $[gh]$ and $[hg]$ are complex conjugates. However, this exception tends to prove the rule: while he refers for one particular case to Hermite's 1855 article (Hermite, 1855a), Christoffel does not attribute the general construction to Hermite nor does he give any special name to these bilinear forms. In any case, Hermitian forms, under any name, reappear on the mathematical scene in the 1880s after their eclipse in the 1860s and 1870s.

In our restricted range of cases, there are also perceptible differences in the way Hermitian forms are handled and perhaps perceived in this half century. In Hermite's work, they appear as particular cases of quadratic forms (with twice as many indeterminates) and parallels are drawn to other types of quadratic forms, similarly distinguished inside all quadratic forms by the specific transformations which leave them invariant. Later, we have seen that they are studied either with ordinary quadratic forms, or in parallel with Dirichlet quadratic forms, depending on the domain. And at the very end of the period, on the contrary, quadratic forms may be treated as a particular case of Hermitian forms, in which complex conjugation reduces to identity.

Let us come back to the question of long-term history, or in this particular case, to a history of Hermitian forms in the second half of the nineteenth century. It is clear that it should integrate the various episodes I just outlined, take into account the breaks we found and establish some continuities between these episodes, that is understand how they are related. Links are perhaps the easiest to trace, in a variety of ways. Some are quite tenuous, for instance the terminology, or the way complex conjugation is denoted, but it may indicate which sources have been used.

One can also of course use cross-references among articles. For instance, Picard's work on hyperfuchsian groups and differential equations inspires Fubini, even if his papers on the future Fubini-Study metric do not (yet) touch the same subjects. But other links extend on a much longer duration: the problem of the transformation of a quadratic form in itself crosses our story many times, from Borchardt's work to Loewy's and beyond. One might also think of Cayley's definition of a metric in a projective space, linked to his work on invariants in the 1860s, and which will be, mostly mediated by Klein, a model for the formulas used in the complex case at the end of the century.

It is important in this respect to analyze these links as concretely as possible: if the same topic (for intance the transformation of a quadratic form into itself) appears in a series of papers published regularly over a certain period, elaborating on each other, we may speak of continuity. But an older reference in several papers, such as Hermite's 1854 paper returning to the forefront in the 1880s, does not say much about the transmission of his ideas. In this case, for instance, Hermite's name and paper are often cited because Picard cites them, and not read directly. And the personal proximity and well-documented communications between Hermite himself and Picard (his son in law), or Jordan's good knowledge of Hermite's work, may explain this resurrection of Hermitian forms in the 1880s. Obviously, links that may be useful for tracing the development of mathematics are not always displayed in publications. For instance, in November 1894, Segre wrote to Adolf Hurwitz (Brigaglia, 2016, 276):

> I take this opportunity to draw your attention to my notes (which you have) entitled "Un nuovo campo di ricerche geom." and "Le rappresentaz. reali delle forme complesse..." because if you, continuing your arithmetical research, pass to the forms of Hermite, you will perhaps find some points of contact with my work. In fact, I study there, among other hyperalgebraic entities, those that I call hyperconics, etc, which are represented analytically by the forms of Hermite [...] I am persuaded that profit can be drawn from studying arithmetical questions with geometrical aids.

As for discontinuiites, we have yet few studies focussing specifically on these issues, that is how to describe as precisely as possible breaks in the mathematical development. It may happen that different aspects change simultaneously and we may tend to speak in this case of a "revolution" (Gillies, 1992). Very often however this is not the case (Gilain and Guilbaud, 2015). Indifference is of course the most obvious cause of decline, but rediscovery or recycling into another theory are quite frequent in modern times and the circumstances of such rebirth are often puzzling.

We might think of invariant theory itself, given over for dead (Fisher, 1966), but then born again, "like an Arabian phoenix arising from its ashes" (Rota, 2001). Or the so-called "theory of order," that Louis Poinsot, inspired by the relations among roots of unity, promoted at the beginning of the century, and which disappeared and reappeared (still attached to Poinsot's name) several times, in mathematics and also in ornamental architecture (Boucard and Eckes, 2015). Another example is tactics, a field which mixed what we would now describe as combinatorics and group theory (Ehrhardt, 2015). A different type of (local) discontinuity, at a semiotical level, has been detected and analyzed by Alain Herreman (Herreman, 2013): in what he calls "inaugural texts," objects which had already appeared informally in mathematics receive a formal definition and acquire, so to speak, a mathematical citizenship.

In other situations, however, it is necessary to change one's scale of observation in order to understand the discontinuity. In our case, for instance, the role of complex numbers in mathematics may be a decisive factor. Another major change has to be taken into account: what is to be considered to be algebra and what it is expected to do. At mid-century, algebra was the theory of equations and forms, and the beginnings of invariant theory, particularly determinants. The program of classification and reduction of equations and forms generated various concepts: characteristic polynomials and their roots, which we have met several times, linear transformations, matrices, etc. (Hawkins, 1977; 1987; Brechenmacher, 2010; Hawkins, 2013). That the "secular equation," which describes the long-term perturbations of planetary motion (our characteristic polynomial for a symmetric matrix), has only real roots was an incentive to work on the complex case for Alfred Clebsch (Clebsch, 1860) and Christoffel (Christoffel, 1864), whom I have mentioned above: it had indeed interesting physical applications. More generally, the questions and results of this development would be decisive for most of the articles of the 1880s and 1890s I have mentioned: most of them prove at one point the properties of the characteristic equation for bilinear and Hermitian forms, or rely on them. But it seems to have partially severed the links to Hermite's special quadratic forms for a while.

In this respect, a striking testimony of change and continuity is given by Max Born's 1914 study of the diamond (Born, 1914): secular equations appeared here again, a long time thread linking mathematical and physical interests. To express them in a simple way, Born used a determinant with complex elements, such that two elements symmetrical with respect to the main diagonal are complex conjugates, thus is "real despite its complex appearance." But in 1914, he had no more difficulty to acknowledge in a footnote that this determinant thus derives from a Hermitian form.

In general history, following Fernand Braudel's suggestion to take into account not only the short-term events of political life, but also phenomena deployed over a more extended period of time, be they economic trends or even geological transformations, the "long term" has come to mean a time of immobility with respect to human actions. But it is not the case in mathematics. A long term thread, based on a simple retrospective identification of concepts, or results, would be a snare. There is no easy line from Hermite to Picard to Moore and Study to Born. Reading Poincaré's sentence again, "art," not "same," should be emphasised. The links we have seen, whatever their lengths in time, are constructed through the "art," the work of mathematicians; identity is the trace of these links. Long term histories require to study not only the local configurations but also the various ways in which they are, or not, connected. This involves the work of mathematicians and as such is—or should be—a primary concern for the history of mathematics.

References

A s, L . 2011. *Étienne Bézout (1730-1783), mathématicien des Lumières*, L'H , s.

A , . 1989. *Physics as a constraint on mathematical research: the case of potential theory and electrodynamics, 1840-1880*, H s M M s, . 28–75.

———. 2011. *Di erential Equations and Algebraic Transcendents*, **17 (2)**, 373–401.

A, M 1991. *Ethnomathematics: A Multicultural View of Mathematical Ideas*, B /C , fi G .

A, K (.) 1984. *History and Philosophy of Modern Mathematics*, M , M .

A , D A D D . 2002. *Writing the History of Dynamical Systems and Chaos: L D and Revolution, Disciplines and Cultures*, H M **29**, 1–67.

A , D C G (.) 2014. *A War of Guns and Mathematics: Mathematical Practices and Communities in France and Its Western Allies around World War I*, A M , .I.

A , E M H (.) 1993. *Messengers of Mathematics: European Mathematical Journals (1800-1946)*, I E E , M .

B , L . 1891. *Geometrische Darstellung der Gruppen linearer Substitutionen mit ganzen complexen Coe cienten nebst Anwendungen auf die Zahlentheorie*, M A **38**.

B , M . 1914. *Zum Raumgittertheorie des Diamanten*, A **44**, 605–642.

B , M J . 1925. *Zur Quantenmechanik*, **34(1)**, 858–888.

B , . 1994. *Va' pensiero: Immagini della matematica nell'Italia dell'Ottocento*, I M B

B , . 2013. *La patria ci vuole eroi: Matematici e vita politica nell'Italia del Risorgimento*, , B

B , J C E . 2015. *La théorie de l'ordre de Poinsot à Bourgoin : mathématiques, philosophie, art ornemental*, **136**, 403–447.

B , F . 2007. *L'identité algébrique d'une pratique portée par la discussion sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des planètes (1766–1874)*, **1**, 5–85.

———. 2010. *Une histoire de l'universalité des matrices mathématiques*, **131**, . 4, 569–603.

———. 2016. *Algebraic Generality versus Arithmetic Generality in the Controversy between C. Jordan and L. Kronecker (1874)*, H G M . 433–467.

B , . 1994. *La Mesure de l'État: administrateurs et géomètres au XVIII^e siècle*, A M , .

B A . 2007. *An Overview of Italian Arithmetic after the* D A , A C. G. G 's *Disquisitiones Arithmeticae*, . 431–452.

———. 2016. *Segre and the foundations of geometry: From complex projective geometry to dual numbers*, F C M A G , . 265–288.

C , G , A C , L G , L G , M M , A 2016. *From Classical to Modern Algebraic Geometry: Corrado Segre's Mastership and Legacy*, B , B .

C , A . 1866. *A supplementary memoir on the theory of matrices*, L **156**, 25–35.

C K , C , D (.) 2016. *The Oxford Handbook of Generality in Mathematics and the Sciences*, .

C , . 2010. *From Problems to Structures: The Cousin Problems and the Emergence of the Sheaf Concept*, A H E **64(1)**, 1–73.

C ff , E B . 1864. *Verallgemeinerung einiger Theoreme des Herrn Weierstrass*, J M **63**, 255–272.

C , A . 1860. *Theorie der circularpolarisirenden Medien*, J M **57**, 319–358.

C (.) 1982. *Celebrazione del centenario della nascita di Guido Fubini e Francesco Severi: Torino, 8-10 Ottobre, 1979*, A , .

C , L . 2004. *Modern Algebra and the Rise of Mathematical Structures*, 2 ., B , B .

C , . 2006. *Arthur Cayley, Mathematician Laureate of the Victorian Age*, J H , B .

’A [illegible], [illegible]. 2000. *Historiographical Proposal for Non-Western Mathematics*, M[illegible]-[illegible] A[illegible] C[illegible]: H[illegible]-[illegible] M[illegible], [illegible]. 79–92.
D[illegible] M[illegible], L[illegible] M[illegible] B[illegible]. 2008.

——— (.) 1999. *The Symbolic Universe: Geometry and Physics, 1890-1930, 1999*,

———. 2000. *Linear Di erential Equations and Group Theory from Riemann to Poincaré*, B , B .

H , . 2005. *Eduard Study (1862-1930): Ein mathematischer Mephistopheles im geometrischen Gärtchen*, D

H . 1977. *Weierstrass and the Theory of Matrices*, A H E **17(2)**, 119–163.

———. 1987. *Cayley's Counting Problem and the Representation of Lie Algebras*, I C M , A 3-11, 1986, . 1642–1656.

———. 2013. *The Mathematics of Frobenius in Context : a journey through 18th to 20th century mathematics*, , .

H , C . 1848. *Note sur un théorème relatif aux nombres entiers*, J **13**, 15.

———. 1854. *Sur la théorie des formes quadratiques (second mémoire)*, J M **47**, 340–368. . . 1, . G - , . 234–263.

———. 1855. *Remarque sur un théorème de M. Cauchy*, C 'A **41**, 181–183.

———. 1855 . *Sur la théorie de la transformation des fonctions abéliennes*, C 'A **40**, 249–254, 304–309, 365–369, 427–431, 485–489, 536–541, 704–707, 784–787.

———. 1856. *Sur le nombre des racines d'une équation algébrique comprises entre des limites données*, J M **52**, 39–51.

H , C . 1905. *Correspondance*, G - , .

H , A . 2013. *L'inauguration des séries trigonométriques dans la de Fourier et dans la controverse des cordes vibrantes*, ' - **19(2)**, 151–243.

H , A . 1996. *Sur la recomposition du paysage mathématique japonais au début de l'époque meiji*, M E : , , , . 247–268.

H , C . 1991. *Aux origines de la géométrie algébrique : les travaux de Picard sur les surfaces (1884-1905)*, L F .

K , A G (.) 2014. *Handbook on the History of Mathematics Education*, , .

K , H ff. 2002. *Di erent motivations and goals in the historical development of the theory of systems of linear inequalities*, A H E **56**, 469–538.

———. 2010. *History of convexity and mathematical programming: Connections and relationships in two episodes of research in pure and applied mathematics of the 20th century*, - I C M , H 2010, . 3233–3257.

K , F . 1875.

M , H 1990. *Moderne Sprache Mathematik*, , F
M , F . 1899. *Invariantentheorie*, E , . 320–403.
M , G A., H F. B L E. D . 1916. *Theory and Applications of Finite Groups*, L .
M , E H . 1898. *An Universal Invariant for Finite Groups of Linear Substitutions: with Application in the Theory of the Canonical Form of a Linear Substitution of Finite Period*, M A **50**, 213–219.
M , J , M G (.) 2012. *Diagrams in Mathematics: History and Philosophy, special issue*, . 186-1, .
, . 1999. *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History*, C , C .
, K . 1989. *Toward a History of Nineteenth-Century Invariant Theory*, H M M , . 157–206.
______. 1995. *Mathematics in National Contexts (18751900): An International Overview*, - I C M , A -11, 1994, . 1581–1591.
______. 2006. *John Sylvester, Jewish Mathematician of a Victorian World*, J H , B .
, K D . 1994. *The Emergence of the American Mathematical Research Community 1876-1900*, AM LM , (.I.)
ff , J . 1998. *Faire des mathématiques par lettres*, ' **4**, 143–157.
, E . 1881. *Sur un théorème relatif aux surfaces pour lesquelles les coordonnées d'un point quelconque s'expriment par des fonctions abéliennes de deux paramètres*, M A **19**, 569–577.
______. 1882 *Sur certaines fonctions uniformes de deux variables, et sur un groupe de substitutions linéaires*, C 'A **94**, 579–582.
______. 1882 . *Sur certaines formes quadratiques et sur quelques groupes discontinus*, C 'A **95**, 763–766.
______. 1884. *Mémoire sur les formes quadratiques binaires indéfinies à indéterminées conjuguées*, A 'E **1**, 9–54.
______. 1887. *Remarque sur les groupes linéaires d'ordre fini à trois variables*, B F **15**, 152–156.
______. 1889. *Notice sur les travaux scientifiques*, G - , .
, H . 1908. *Science et méthode*, F , .
. 2010. *Eine Disziplin und ihre Verleger: Disziplinenkultur und Publikationswesen der Mathematik in Deutschland, 1871–1949*, B .
, H -J . 1994. *Experimental Systems: Historiality, Narration, and Deconstruction*, C **7(1)**, 65–81.
, J . 2004. *Reading Strasbourg 368: A Thrice-Told Tale*, H , H . 177–200.
G -C . 2001. *What is invariant theory, really?*, A C C - , . 41–56.
, D (.) 1992. *David Hilbert: Natur und Erkenntniss*, B , B .
______. 2003. *Mathematical Schools, Communities, and Networks*, C H , . 113–132.
2010. *Rewriting points*, I C M - H 2010, . 3258–3291.
, K -H M (.) 2011. *Mathematics meets physics: A contribution to their interaction in the 19th and the first half of the 20th century*, H D , F -M .
, G 1985 *Das mathematische Seminar der Universität Münster, 1831/1875 bis 1951*, ff A **69**, 154–191.
______. 1985 . *Die Entwicklung des mathematischen Seminars der Universität Bonn, 1864-1929*, J M - **87**, 139–163.

———. 2005. *Conflicts Between Generalization, Rigor, and Intuition: Number Concepts Underlying the Development of Analysis in 17th-19th Century France and Germany*, ,
.
, C . 1890. *Un nuovo campo di ricerche geometriche*, A A
25, 276–301, 430–457.
———. 1890 . *Un nuovo campo di ricerche geometriche*, A A
26, 35–71, 592–612.
, H 1994. *Corps et modèles*, , .
, E . 1905. *Kürzeste Wege im komplexen Gebiet*, M A **60**, 321–378.
, A . 1926. *Corrado Segre (1863–1924)*, J DM **35**, 209–250.
, D ( ) 2011. *History of Numerical and Graphical Tables*,
. 8.
, D . 2012. *Diagrams in the theory of di erential equations (eighteenth to nineteenth centuries)*, **186**, 257–288.
, K . 1900. *Arithmetische Theorie der Formen*, E
, . 581–634.
, H . 1889. *De endelige Transformations-gruppers Theori*, B L , C -
.
, 2009. *Le J L et la presse de son temps : une entreprise dédition et de circulation des mathématiques au XIXème siècle (1824–1835)*,
, A . 1980. *History of mathematics: why and how*, I C -
M 1978, . 227–236.

CNRS, Institut de mathématiques de Jussieu-Paris Rive Gauche, UMR 7586, Sorbonne Université, UPMC, Univ Paris 06, Univ Paris Diderot, Sorbonne Paris Cité, Case 247, 4 place Jussieu, 75252 Paris Cedex, France

E-mail address: catherine.goldstein@imj-prg.fr