Calculus, heat flow and curvature-dimension bounds in metric measure spaces

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1 Introduction

The theory of curvature-dimension bounds for nonsmooth spaces has several motivations: the study of functional and geometric inequalities in structures which are very far from being Euclidean, therefore with new non-Riemannian tools, the description of the "closure" of classes of Riemannian manifolds under suitable geometric constraints, the stability of analytic and geometric properties of spaces (e.g. to prove rigidity results). Even though these goals may occasionally be in conflict, in the last few years we have seen spectacular developments in all these directions, and my text is meant both as a survey and as an introduction to this quickly developing research field.

I will mostly focus on *metric measure spaces* (m.m.s. in brief), namely triples $(X, \mathsf{d}, \mathfrak{m})$, where (X, d) is a complete and separable metric space and $\mathfrak{m}$ is a non-negative Borel measure, finite on bounded sets, typically with $\operatorname{supp}\mathfrak{m} = X$. The model case that should always be kept in mind is a *weighted* Riemannian manifold $(\mathbb{M}, \mathrm{g}, \mathfrak{m})$, with $\mathfrak{m}$ given by

$$\mathfrak{m} := e^{-V}\mathrm{vol}_{\mathrm{g}} \tag{1.1}$$

for a suitable weight function $V : \mathbb{M} \to \mathbb{R}$. It can be viewed as a metric measure space by taking as $\mathsf{d} = \mathsf{d}_{\mathrm{g}}$ the Riemannian distance induced by g.

In order to achieve the goals I mentioned before, it is often necessary to extend many basic calculus tools from smooth to nonsmooth structures. Because of this I have organized the text by starting with a presentation of these tools: even though some new developments of calculus in m.m.s. have been motivated by the theory of curvature-dimension bounds, the validity of many basic results does not depend on curvature and it is surely of more general interest. In this regard, particularly relevant are results which provide a bridge between the so-called "Eulerian" point of view (when dealing with gradients, Laplacians, Hessians, etc.) and the so-called "Lagrangian" point of view (when dealing with curves in the ambient space). In the theory of curvature-dimension bounds, these bridges are crucial to connect the Lott-Villani and Sturm theory, based on Optimal Transport (therefore Lagrangian) to the Bakry-Émery theory, based on Γ-calculus (therefore Eulerian), in many cases of interest.

The limitation on the length of this text forced me to make difficult and subjective choices, concerning both references and topics; for this reason and not for their lack of importance I will not mention closely related areas of investigation, such as the many variants and regularizations of optimal transport distances, curvature-dimension bounds in sub-Riemannian structures, rigidity results, time-dependent metric measure structures, and others.

2 Calculus tools in metric spaces

Let us start with some basic tools and terminology, at the metric level. Recall that a curve $\gamma : [0, T] \to X$ is said to be *absolutely continuous* if there exists $g \in L^1(0, T)$ satisfying

$$\mathsf{d}(\gamma_s, \gamma_t) \le \int_s^t g(r)\, dr \qquad \forall 0 \le s \le t \le T.$$

Among absolutely continuous curves, Lipschitz curves play a special role. Among them, we shall denote by $\mathrm{Geo}(X)$ the class of *constant speed geodesics* $\gamma : [0, 1] \to X$, characterized by

$$\mathsf{d}(\gamma_s, \gamma_t) = |s - t|\mathsf{d}(\gamma_1, \gamma_0) \qquad \forall s,\, t \in [0, 1].$$

A metric space (X, d) is said to be *geodesic* if any pair of points can be connected by at least one $\gamma \in \mathrm{Geo}(X)$.

In this survey, K*-convex* functions, with $K \in \mathbb{R}$, play an important role. In the smooth setting, K-convexity corresponds to the lower bound $\mathrm{Hess}\, f \ge K\, \mathrm{Id}$ on the Hessian of f, but the definition is immediately adapted to the metric setting, by requiring that $f \circ \gamma$ is K-convex (i.e. $t \mapsto f(\gamma_t) - \frac{1}{2}Kt^2\mathsf{d}^2(\gamma_0, \gamma_1)$ is convex in $[0, 1]$) for all $\gamma \in \mathrm{Geo}(X)$.

Definition 2.1 (Metric derivative)**.** Let $\gamma : [0, T] \to X$ be absolutely continuous. Then, it can be proved that for $\mathscr{L}^1$-a.e. $t \in (0, T)$ the limit

$$|\gamma'|(t) := \lim_{h \to 0} \frac{\mathsf{d}(\gamma_{t+h}, \gamma_t)}{|h|}$$

exists. We call this limit metric derivative: it is indeed the minimal function $g \in L^1(0, T)$, up to $\mathscr{L}^1$-negligible sets, such that the inequality $\mathsf{d}(\gamma_s, \gamma_t) \le \int_s^t g(r)\, dr$ holds for all $0 \le s \le t \le T$.

Building on this definition, one can define the space of curves $\mathrm{AC}^p([0, T]; X)$, $1 \le p \le \infty$, by requiring p-integrability of the metric derivative. Also, as in

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the smooth setting, the metric derivative provides an integral representation to the curvilinear integrals

$$\int_\gamma g\,d\sigma := \int_0^T g(\gamma_s)|\gamma'|(s)\,ds = \int g\,dJ\gamma \quad \text{with } J\gamma := \gamma_\#(|\gamma'|\mathscr{L}^1) \tag{2.1}$$

which otherwise should be defined using integration on $\gamma([0,T])$ w.r.t. the 1-dimensional Hausdorff measure $\mathscr{H}^1$ (counting multiplicities if γ is not 1-1). In turn, the inequality

$$|f(\gamma_1) - f(\gamma_0)| \le \int_\gamma g\,d\sigma, \tag{2.2}$$

valid with $g = |\nabla f|$ in a smooth setting, leads to the notion of *upper gradient* [68].

Definition 2.2 (Upper gradient). We say that a Borel function $g : X \to [0,\infty]$ is an upper gradient of $f : X \to \mathbb{R}$ if the inequality (2.2) holds for any $\gamma \in \mathrm{AC}([0,1];X)$.

Clearly the upper gradient should be thought of as an upper bound for the modulus of the gradient of f[1]. Without appealing to curves, the "dual" notion of *slope* (also called local Lipschitz constant) simply deals with difference quotients:

Definition 2.3 (Slope). For $f : X \to \mathbb{R}$ the slope $|\nabla f|(x)$ of f at a non-isolated point $x \in X$ is defined by

$$|\nabla f|(x) := \limsup_{y\to x} \frac{|f(y)-f(x)|}{\mathsf{d}(y,x)}.$$

It is simple to check that the slope is an upper gradient for Lipschitz functions. In the theory of metric gradient flows a key role is also played by the *descending* slope, a one-sided counterpart of the slope:

$$|\nabla^- f|(x) := \limsup_{y\to x} \frac{\max\{f(x)-f(y),0\}}{\mathsf{d}(x,y)}. \tag{2.3}$$

The notion of gradient flow, closely linked to the theory of semigroups, also plays an important role. If we are given a K-convex and lower semicontinuous function $F : X \to (-\infty,\infty]$, with X Hilbert space, the theory of evolution problems for maximal monotone operators (see for instance

[1]Strictly speaking it should be the modulus of the differential, the natural object in duality with curves, but the "gradient" terminology is by now too estabilished to be changed. However, as emphasized in [54, Sec 3], this distinction is crucial for the development of a good theory.

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[30]) provides for any $\bar{x} \in \overline{\{F < \infty\}}$ a locally absolutely continuous map $x_t : (0, \infty) \to X$ satisfying

$$\frac{d}{dt} x_t \in -\partial_K F(x_t) \text{ for } \mathscr{L}^1\text{-a.e. } t \in (0, \infty), \qquad \lim_{t \to 0} x_t = \bar{x}, \tag{2.4}$$

where $\partial_K F$ stands for the K-subdifferential of F, namely

$$\partial_K F(x) := \left\{ \xi \in X : \ F(y) \geq F(x) + \langle \xi, y - x \rangle + \frac{K}{2} |y - x|^2 \ \forall y \in X \right\}.$$

Besides uniqueness, a remarkable property of the gradient flow x_t is a selection principle, which turns the differential inclusion into an equation: for $\mathscr{L}^1$-a.e. $t \in (0, \infty)$ one has that $-\frac{d}{dt} x_t$ is the element with minimal norm in $\partial_K F(x_t)$. Moreover, differentiating the square of the Hilbert norm one can write (2.4) in an equivalent form, called *Evolution Variational Inequality* (in short EVI_K)

$$\frac{d}{dt} \frac{1}{2} |x_t - y|^2 \leq F(y) - F(x_t) - \frac{K}{2} |y - x_t|^2 \quad \mathscr{L}^1\text{-a.e. in } (0, \infty), \text{ for all } y \in X. \tag{2.5}$$

This way, the scalar product does not appear anymore and this formulation, involving energy and distance only, makes sense even in metric spaces.

We conclude this section by recalling the *metric* notion of gradient flow, based on a deep intuition of E.De Giorgi, see [7] for much more on this subject. Assume for the moment that we are in a smooth setting (say F of class C^1 on a Hilbert space X). Then, we can encode the *system* of ODE's $\gamma' = -\nabla F(\gamma)$ into a *single* differential inequality: $-2(F \circ \gamma)' \geq |\gamma'|^2 + |\nabla F|^2(\gamma)$. Indeed, for any $\gamma \in C^1$ one has

$$-2(F \circ \gamma)' = -2\langle \nabla F(\gamma), \gamma' \rangle \leq 2|\nabla F|(\gamma)|\gamma'| \leq |\gamma'|^2 + |\nabla F|^2(\gamma).$$

Now, the first inequality is an equality iff $\nabla F(\gamma)$ is parallel to $-\gamma'$, while the second one is an equality iff $|\nabla F|(\gamma) = |\gamma'|$, so that by requiring the validity of the converse inequalities we are encoding the full ODE. In the metric setting, using metric derivatives and the descending slope (2.3) and moving to an integral formulation of the differential inequality, this leads to the following definition:

Definition 2.4 (Metric gradient flow). Let $F : X \to (-\infty, \infty]$ and $\bar{x} \in \{F < \infty\}$. We say that a locally absolutely continuous curve $\gamma : [0, \infty) \to X$ is a metric gradient flow of F starting from $\bar{x}$ if

$$F(\gamma_t) + \int_0^t \frac{1}{2} |\gamma'|^2(r) + \frac{1}{2} |\nabla^- F|^2(\gamma_r) \, dr \leq F(\bar{x}) \qquad \forall t \geq 0. \tag{2.6}$$

Under the assumption that $|\nabla^- F|$ is an upper gradient of F (this happens for Lipschitz functions or, in geodesic spaces, for K-convex functions)

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one obtains that equality holds in (2.6), that $t \mapsto F(\gamma_t)$ is absolutely continuous in $[0, \infty)$, and that $|\gamma'| = |\nabla^- F|(\gamma)$ $\mathscr{L}^1$-a.e. in $(0, \infty)$. Reasoning along these lines one can prove that, for K-convex and lower semicontinuous functions in Hilbert spaces, the metric and differential notions of gradient flow coincide. However, in general metric spaces the existence of an EVI_K-flow is a much stronger requirement than the simple energy-dissipation identity (2.6): it encodes not only the K-convexity of Φ (this has been rigorously proved in [44]) but also, heuristically, some infinitesimally Hilbertian behaviour of d.

3 Three basic equivalence results

Curvature conditions deal with second-order derivatives, even though often - as happens for convexity - their synthetic formulation at least initially involves difference quotients or first-order derivatives. Before coming to the discussion of synthetic curvature conditions, in this section I wish to describe three basic equivalence results at the level of "first order differential calculus" (weakly differentiable functions, flow of vector fields, metric versus energy structures), which illustrate well the Eulerian-Lagrangian duality I mentioned in the introduction.

3.1 Cheeger energy and weakly differentiable functions

The theory of weakly differentiable functions, before reaching its modern form developed along different paths, with seminal contributions by B.Levi, J.Leray, L.Tonelli, C.B.Morrey, G.C.Evans, S.L.Sobolev (see [82] for a nice historical account). In Euclidean spaces, we now recognize that three approaches are essentially equivalent: approximation by smooth functions, distributional derivatives and of good behaviour along almost all lines. More surprisingly, this equivalence persists even in general metric measure structures. In what follows, I will restrict my discussion to the case of p-integrable derivatives with $1 < p < \infty$; in the limiting case $p = 1$ the results are weaker, while for BV functions the full equivalence still persists, see [3, 76, 16].

To illustrate this equivalence, let me start from the approximation with smooth functions, now replaced by Lipschitz functions in the m.m.s. category. The following definition is inspired by Cheeger's [36], who dealt with a larger class of approximating functions (the functions with p-integrable upper gradient), see also [38, Appendix 2].

Defnition 3.1 ($H^{1,p}$ Sobolev space)**.** We say that $f \in L^p(X, \mathfrak{m})$ belongs to $H^{1,p}(X, \mathsf{d}, \mathfrak{m})$ if there exist a sequence $(f_i) \subset \mathrm{Lip}_b(X, \mathsf{d})$ with $f_i \to f$ in L^p and $\sup_i \| |\nabla f_i| \|_p < \infty$.

This definition is also closely related to the so-called *Cheeger energy*

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$\mathrm{Ch}_p : L^p(X, \mathfrak{m}) \to [0, \infty]$, namely

$$\mathrm{Ch}_p(f) := \inf \left\{ \liminf_{i\to\infty} \int_X |\nabla f_i|^p \, d\mathfrak{m} : \ f_i \to f \text{ in } L^p(X, \mathfrak{m}), \ f_i \in \mathrm{Lip}_b(X, \mathsf{d}) \right\}, \tag{3.1}$$

which turns out to be a convex and $L^p(X, \mathfrak{m})$-lower semicontinuous functional, whose finiteness domain coincides with $H^{1,p}$ and is dense in L^p. Then, by looking for the optimal approximation in (3.1), J.Cheeger identified a distinguished object, the *minimal relaxed slope*, denoted $|\nabla f|_*$: it provides the integral representation

$$\mathrm{Ch}_p(f) = \int_X |\nabla f|_*^p \, d\mathfrak{m} \qquad \forall f \in H^{1,p}(X, \mathsf{d}, \mathfrak{m})$$

which corresponds, in the smooth setting and for $p = 2$, to the *weighted Dirichlet energy* $\int_M |\nabla f|^2 e^{-V} \, d\mathrm{vol}_{\mathrm{g}}$.

Even at this high level of generality one can then estabilish basic calculus rules, such as the chain rule. In addition, $f \mapsto |\nabla f|_*$ has strong locality properties, which pave the way to connections with the theory of Dirichlet forms, when $p = 2$ and Ch_p is a quadratic form.

The convexity and lower semicontinuity of Ch_p allow us, when $p = 2$, to apply the well-estabilished theory of gradient flows in Hilbert spaces to provide, for all $\bar f \in L^2(X, \mathfrak{m})$ the unique gradient flow of $\frac{1}{2}\mathrm{Ch}_2$ starting from $\bar f$. In addition, the selection principle of the Hilbertian theory of gradient flows motivates the following definition and terminology, consistent with the classical setting.

Definition 3.2 (Laplacian Δ and Heat flow P_t). Let $g \in L^2(X, \mathfrak{m})$ be such that $\partial_0 \mathrm{Ch}_2(g)$ is not empty. We call Laplacian of g, and denote Δg, the element with minimal norm in $-\frac{1}{2}\partial_0 \mathrm{Ch}_2(g)$. With this notation, for all $f \in L^2(X, \mathfrak{m})$ we denote by $\mathsf{P}_t f$ the unique solution to (2.4) with $F = \frac{1}{2}\mathrm{Ch}_2$, thus solving the equation

$$\frac{d}{dt}\mathsf{P}_t f = \Delta \mathsf{P}_t f \ \text{ for } \mathscr{L}^1\text{-a.e. } t \in (0, \infty).$$

Notice that Δ, also called weighted Laplacian or Witten Laplacian in the smooth context, depends *both* on d and $\mathfrak{m}$: this can be immediately understood in the setting of weighted Riemannian manifolds, since ∇f depends on d (i.e. the Riemannian metric g) but the divergence, viewed as adjoint of the gradient operator in $L^2(X, \mathfrak{m})$, depends on $\mathfrak{m}$, so that

$$\Delta f = \Delta_{\mathrm{g}} f - \langle \nabla V, \nabla f \rangle \tag{3.2}$$

and Δ reduces to the Laplace-Beltrami operator Δ_{g} when $V \equiv 0$ in (1.1).

By the specific properties of Ch_2, under the global assumption (5.11) the semigroup P_t can also be extended to a semigroup of contractions in every

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$L^p(X,\mathfrak{m})$, $1 \le p \le \infty$, preserving positivity, mass, and constants. We retain the same notation P_t for this extension.

As simple examples illustrate (see Section 3.2), Ch_2 need not be a quadratic form, so that in general neither the operator Δ nor the semigroup P_t are linear; nevertheless basic calculus rules and differential inequalities still apply, see [9, 54].

Coming back to our discussions about Sobolev spaces, one can try to define/characterize weakly differentiable functions by appealing to the behaviour of functions along lines (and, in a nonsmooth setting, curves). Actually, this is the very first approach to the theory of weakly differentiable functions, pioneered by B.Levi in 1906 [71] in his efforts to put the Dirichlet principle on firm grounds. Later on, it was revisited and, at the same time, made frame-indifferent by B.Fuglede [50] in this form: $f : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ belongs to the Beppo Levi space if, for some vector $\nabla f \in L^p(\Omega;\mathbb{R}^N)$, one has

$$f(\gamma_1) - f(\gamma_0) = \int_\gamma \nabla f \qquad \text{for } \mathrm{Mod}_p\text{-almost every } \gamma.$$

Here Fuglede used a potential-theoretic notion, the so-called p-Modulus: for a family Γ of (non parametric) curves, in $\mathbb{R}^N$, one defines

$$\mathrm{Mod}_p(\Gamma) := \inf\left\{\int_{\mathbb{R}^N} \rho^p \, d\mathfrak{m} : \int_\gamma \rho \, d\sigma \ge 1 \ \ \forall \gamma \in \Gamma\right\}. \tag{3.3}$$

In more recent times, N.Shanmughalingam [96] adapted this concept to the metric measure setting, with the introduction of the *Newtonian space* $N^{1,p}(X,\mathsf{d},\mathfrak{m})$; notice that the notion of p-Modulus immediately extends to the metric measure setting, understanding curvilinear integrals as in (2.1).

Definition 3.3 ($N^{1,p}$ Sobolev space). We say that $f \in L^p(X,\mathfrak{m})$ belongs to $N^{1,p}(X,\mathsf{d},\mathfrak{m})$ if there exist $\tilde f : X \to \mathbb{R}$ and $g \in L^p(X,\mathfrak{m})$ non-negative such that $\tilde f = f$ $\mathfrak{m}$-a.e. in X and $|\tilde f(\gamma_1) - \tilde f(\gamma_0)| \le \int_\gamma g \, d\sigma$ holds for Mod_p-almost every curve γ.

Even in this case one can identify a distinguished object playing the role of the modulus of the gradient, namely the g with smallest L^p-norm among those satisfying $|\tilde f(\gamma_1) - \tilde f(\gamma_0)| \le \int_\gamma g \, d\sigma$ Mod_p-a.e.: it is called *minimal p-weak upper gradient*, and denoted by $|\nabla f|_w$. This point of view has been deeply investigated by the Finnish school, covering also vector-valued functions and the relation with the original H Sobolev spaces of [36], see the recent monographs [27], [61].

Having in mind the theory in Euclidean spaces, we might look for analogues in the metric measure setting of the classical point of view of weak derivatives, within the theory of distributions. I will describe this last point of view, even though for the moment it does not play a substantial role in the theory of curvature-dimension bounds for m.m.s. On a Riemannian

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manifold $(\mathbb{M}, \mathrm{g})$, with $\mathfrak{m} = \mathrm{Vol}_{\mathbb{M}}$, it is natural to define the weak gradient ∇f by the integration by parts formula

$$\int \mathrm{g}(\nabla f, \boldsymbol{b})\, d\mathfrak{m} = -\int f \operatorname{div} \boldsymbol{b}\, d\mathfrak{m} \tag{3.4}$$

against smooth (say compactly supported) vector fields $\boldsymbol{b}$. In the abstract m.m.s. setting, the role of vector fields is played by *derivations*, first studied in detail by N.Weaver in [107]. Here we adopt a definition close to the one adopted in [107], but using [?] to measure of the size of a derivation.

Defnition 3.4

Finally, let me conclude this "calculus" section with a (necessarily) brief mention of other important technical aspects and research directions.

Test plans. In connection with Theorem 3.6, the inclusion $H^{1,p} \subset N^{1,p}$ is not hard to prove, while the converse requires the construction of a "good" approximation of f by Lipschitz functions, knowing only the behaviour of f along Mod_p-almost all curves. To achieve this goal, in [9, 8] besides nontrivial tools (Hopf-Lax semigroup (4.3), superposition principle, etc, described in the next sections) we also use a new and more "probabilistic" way to describe the exceptional curves. This is encoded in the concept of *test plan*.

Defnition 3.7 (Test plan). We say that $\boldsymbol{\eta} \in \mathscr{P}(C([0,T];X))$ is a p-test plan if it is concentrated on $\mathrm{AC}^q([0,1];X)$, with $q = p/(p-1)$, and there exists $C = C(\boldsymbol{\eta}) \geq 0$ such that

$$(e_t)_\#\boldsymbol{\eta} \leq C\mathfrak{m} \quad \forall t \in [0,1], \qquad \text{where } e_t : C([0,1];X) \to X,\ e_t(\gamma) := \gamma_t.$$

Then, we say that a Borel set $\Gamma \subset C([0,T];X)$ is p-negligible if $\boldsymbol{\eta}(\Gamma) = 0$ for any p-test plan $\boldsymbol{\eta}$. Now, we can say that a function $f \in L^p(X,\mathfrak{m})$ belongs to the Beppo Levi space $BL^{1,p}(X,\mathsf{d},\mathfrak{m})$ if, for some $g \in L^p(X,\mathfrak{m})$, the upper gradient property (2.2) holds for p-almost every curve. Since Mod_p-negligible sets are easily seen to be p-negligile one has the inclusion $N^{1,p} \subset BL^{1,p}$, and with the proof of the equality $BL^{1,p} = H^{1,p}$ we have closed the circle of equivalences.

Test plans are useful not only to describe null sets of curves. They are natural objects in the development of calculus in metric measure spaces (for instance the proof of Theorem 3.9 below deeply relies on this concept), since they induce vector fields, i.e. derivations, via the formula

$$\int_X \boldsymbol{b}_{\boldsymbol{\eta}}(f)\phi\, d\mathfrak{m} := \int\int_0^1 \phi\circ\gamma \frac{d}{dt} f\circ\gamma\, dt\, d\boldsymbol{\eta}(\gamma) \qquad \forall \phi \in L^1(X,\mathfrak{m}),$$

which implicity defines $\boldsymbol{b}_{\boldsymbol{\eta}}$. These connections are further investigated in [54], where the notion of test plan representing the gradient of a function is introduced, see also [95], where an analogous analysis is done with the so-called *Alberti representations* $\int J\gamma\, d\boldsymbol{\eta}(\gamma)$ of $\mathfrak{m}$, with J the measure-valued operator on $\mathrm{AC}([0,1];X)$ defined in (2.1). In addition, along these lines one obtains [4] also a useful "dual" representation of the p-Modulus:

$$(\mathrm{Mod}_p(\Gamma))^{1/p} = \sup\left\{\frac{1}{\|\mathrm{bar}\,\boldsymbol{\eta}\|_q} :\ \boldsymbol{\eta}(\Gamma) = 1,\ \boldsymbol{\eta} \in T_q\right\}, \tag{3.5}$$

where $q = p/(p-1)$ and $T_q \subset \mathscr{P}(C([0,1];X))$ is defined by the property $\int J\gamma\, d\boldsymbol{\eta}(\gamma) \ll \mathfrak{m}$, with density, the barycenter $\mathrm{bar}\,\boldsymbol{\eta}$, in $L^q(X,\mathfrak{m})$.

Diferentiable structures on metric measure spaces. One of the main motivation of the seminal p [

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a suitable version of Rademacher's theorem in m.m.s. Roughly speaking, J.Cheeger proved that in doubling m.m.s. satisfying the Poincaré inequality (the so-called PI spaces, see [61] for much more on this subject) one has a countable Borel atlas X_i and Lipschitz maps $F^i : X \to \mathbb{R}^{N_i}$ with the property that $\sup_i N_i < \infty$ and, for any i and $f \in \mathrm{Lip}_b(X)$, there exist $\alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,N_i}$

If we apply these constructions to the Dirichlet energy on a Riemannian manifold, we see that $\Gamma(f,g)$ corresponds to the scalar product between ∇f and ∇g; in this sense we may think that Lipschitz functions provide a differentiable structure (with global sections of the tangent bundle provided by gradient vector fields) and Dirichlet forms provides a metric structure (via the operator Γ).

We may move from the metric to the "energy" structure in a canonical way, setting $\mathcal{E} = \mathrm{Ch}_2$ if Cheeger's energy Ch_2 is a quadratic form. However, this property of being a quadratic form is far from being true in general: for instance, if we apply Cheeger's construction to the metric measure structure $(\mathbb{R}^N, \mathsf{d}, \mathscr{L}^N)$, where $\mathsf{d}(x,y) = \|x-y\|$ is the distance induced by a norm, we find that $|\nabla f|_* = \|\nabla f\|_*$ for all $f \in H^{1,p}$, where ∇f is the weak (distributional) derivative and $\|\cdot\|_*$ is the dual norm. Hence, the norm is Hilbertian iff Ch_2 is a quadratic form. This motivates the following terminology introduced in [54].

Definition 3.8 (Infinitesimally Hilbertian m.m.s.)**.** We say that a m.m.s. $(X,\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian if Cheeger's energy Ch_2 in (3.1) is a quadratic form in $L^2(X,\mathfrak{m})$.

For infinitesimally Hilbertian m.m.s., the following consistency result has been proved in [10, Thm. 4.18], see also [54].

Theorem 3.9. *If $(X,\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian, then Ch_2 is a strongly local Dirichlet form and its carré du champ $\Gamma(f)$ coincides with $|\nabla f|_*^2$.*

In order to move in the opposite direction, we need to build a distance out of $\mathcal{E}$. The canonical construction starts from the class

$$\mathcal{C} := \{f \in \mathbb{V} \cap C_b(X,\tau) : \ \Gamma(f) \le 1 \ \mathfrak{m}\text{-a.e. in } X\}$$

and defines the *intrinsic distance* by

$$\mathsf{d}_{\mathcal{E}}(x,y) := \sup\{|f(x) - f(y)| : \ f \in \mathcal{C}\}. \tag{3.6}$$

Under the assumption that $\mathcal{C}$ generates a *finite* distance (this is not always the case, as for the Dirichlet form associated to the Wiener space, leading to *extended* metric measure structures [5]) and assuming also that the topology induced by $\mathsf{d}_{\mathcal{E}}$ coincides with τ, we have indeed obtained a metric measure structure. This happens for instance in the classical case of quadratic forms in $L^2(\mathbb{R}^N)$ induced by symmetric, uniformly elliptic and bounded matrices A:

$\mathcal{E}$fi$_A$

these procedures, namely from $\mathcal{E}$ one builds $\mathsf{d}_\mathcal{E}$ and then the Cheeger energy $\mathrm{Ch}_{2,\mathsf{d}_\mathcal{E}}$ induced by $\mathsf{d}_\mathcal{E}$ (or, conversely, one first moves from the metric to the energy structure and then again to the metric structure). To realize that this is a nontrivial issue, I recall what happens at the level of the relation between energy and distance in the case of the quadratic forms $\mathcal{E}_A$ in (3.7): first, even though $\mathcal{E}_A$ is in 1-1 correspondence with A (and this observation is at the basis of the theories of G-convergence for diffusion operators A, and of Γ-convergence), we also know from [101] that $\mathcal{E}_A$ need not be uniquely determined by $\mathsf{d}_{\mathcal{E}_A}$. Moreover, the analysis of the construction in [101] reveals that, given an intrinsic distance d induced by some $\mathcal{E}_B$, there is no "minimal" $\mathcal{E}_A$ whose intrinsic distance is d. Second, an example in [69] shows that, for some $\mathcal{E} = \mathcal{E}_A$, the Cheeger energy $\mathrm{Ch}_{2,\mathsf{d}_\mathcal{E}}$ need not be a quadratic form as in (3.7).

In order to clarify the relations between these objects in the general setting, the following property plays an important role.

Definition 3.10 (τ-upper regularity).

is closely related to the continuity equation

$$\text{(CE)} \qquad \frac{d}{dt}\varrho_t + \mathrm{div}\,(\boldsymbol{b}_t\varrho_t) = 0, \qquad \varrho_0 = \bar\varrho.$$

Indeed, denoting by

$$\boldsymbol{X}(t,x) : [0,T] \times X \to X$$

the flow map of the ODE, under appropriate assumptions, the push-forward measures $\mu_t := \boldsymbol{X}(t,\cdot)_\#(\bar\rho\mathfrak{m})$ are shown to be absolutely continuous w.r.t. $\mathfrak{m}$ and their densities ϱ_t solve the weak formulation of (CE), namely

$$\frac{d}{dt}\int_X \phi\varrho_t\,d\mathfrak{m} = \int_X \langle \boldsymbol{b}_t, \nabla\phi\rangle\varrho_t\,d\mathfrak{m} \tag{3.8}$$

for any test function ϕ (notice that the operator div in (CE), according to (3.4), does depend on the reference measure $\mathfrak{m}$). Under appropriate regularity assumptions (for instance within the Cauchy-Lipschitz theory) one can then prove that this is the unique solution of (CE). Starting from the seminal paper [48], these connections have been extended to classes of nonsmooth (e.g. Sobolev, or even BV [1]) vector fields, with applications to fluid mechanics and to the theory of conservation laws, see the lecture notes [2] for much more information on this topic. One of the basic principles of the theory is that, as I illustrate below, well-posedness can be transferred from the (ODE) to (CE), and conversely.

More recently it has been understood in [18] that not only can one deal with nonsmooth vector fields, but even with general (nonsmooth) metric measure structures. Therefore from now on I come back to this high level of generality. We have already seen that in the m.m.s. setting the role of vector fields is played by derivations, and that the divergence operator can be defined; on the other hand, the definition of solution to the ODE is more subtle. If we forget about the measure structure, looking only at the metric one, there is by now a well-established theory for ODE's $\boldsymbol{b} = -\nabla\mathcal{E}$ of gradient type [7]: in this setting, as we have seen in Section 2, one can characterize the gradient flow by looking at the maximal rate of dissipation of $\mathcal{E}$. In general, for vector fields which are not gradients, one can use *all* Lipschitz functions as "entropies"; taking also into account the role of the measure $\mathfrak{m}$, this leads to the following definition of *regular Lagrangian flow*, an adaptation to the nonsmooth setting of the notion introduced in [1].

Definition 3.12 (Regular Lagrangian Flow). Let $\boldsymbol{b}_t$ be derivations. We say that $\boldsymbol{X}(t,x)$ is a regular Lagrangian flow relative to $\boldsymbol{b}_t$ (in short RLF) if the following three properties hold:

(a) $\boldsymbol{X}(\cdot,x) \in \mathrm{AC}([0,T];X)$ for $\mathfrak{m}$-a.e. $x \in X$;

(b) for all $f \in \mathrm{Lip}_b(X)$ and $\mathfrak{m}$-a.e. $x \in X$, one has $\frac{d}{dt}f(\boldsymbol{X}(t,x)) = \boldsymbol{b}_t(f)(\boldsymbol{X}(t,x))$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$;

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(c) for some $C \geq 0$, one has $\boldsymbol{X}(t,\cdot)_{\#}\mathfrak{m} \leq C\mathfrak{m}$ for all $t \in [0,T]$.

The basic principle of the theory is the following result, reminiscent of the uniqueness in law/pathwise uniqueness results typical of the theory of stochastic processes.

Theorem 3.13. *Assume that* $|\boldsymbol{b}_t| \in L^1((0,T);L^2(\mathfrak{m}))$. *Then* (CE) *is well-posed in the class*

$\mathcal{L} := \Big\{ \varrho \in L^{\infty}$F1301Tf423Td[(2)]TF1001Tf4323Td[(()]TF1001Tf42430Td[(0040012(T(1)]TF00001Tf

Definition 3.15 (Derivations with deformation in L^2)**.** Let $\boldsymbol{b}$ be a derivation in L^2. We write $D^{sym}\boldsymbol{b} \in L^2(X, \mathfrak{m})$ if there exists $c \geq 0$ satisfying

$$\left| \int D^{sym}\boldsymbol{b}(f,g)\, d\mathfrak{m} \right| \leq c\|\Gamma(f)\|_2^{1/2}\|\Gamma(g)\|_2^{1/2}, \tag{3.11}$$

for all f, $g \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ with Δf, $\Delta g \in L^4(X, \mathfrak{m})$, where

$$\int D^{sym}\boldsymbol{b}(f,g)\, d\mathfrak{m} := -\frac{1}{2}\int [\boldsymbol{b}(f)\Delta g + \boldsymbol{b}(g)\Delta f - (\mathrm{div}\, \boldsymbol{b})\Gamma(f,g)]\, d\mathfrak{m}. \tag{3.12}$$

We denote by $\|D^{sym}\boldsymbol{b}\|_2$ be the smallest constant c in (3.11).

Under a mild regularizing property of the semigroup $\mathsf{P_t}$, satisfied for instance in all $\mathrm{RCD}(K,\infty)$ spaces (see [18, Thm. 5.4] for the precise statement), the following result provides well posedness of (CE), and then existence and uniqueness of the RLF $\boldsymbol{X}$, in a quite general setting.

Theorem 3.16. *If*

$$|\boldsymbol{b}_t| \in L^1((0,T); L^2(X,\mathfrak{m})), \quad \|D^{sym}\boldsymbol{b}_t\|_2 \in L^2(0,T), \quad |\mathrm{div}\, \boldsymbol{b}_t| \in L^1((0,T); L^\infty(X,\mathfrak{m})),$$

then (CE) is well posed in the class $\mathcal{L}$ in (3.9).

4 Background on optimal transport

Building on the metric structure (X, d), optimal transport provides a natural way to introduce a geometric distance between probability measures, which reflects well the metric properties of the base space. We call $\mathscr{P}_2(X)$ the space of Borel probability measures with finite quadratic moment, namely μ belongs to $\mathscr{P}_2(X)$ if $\int_X \mathsf{d}^2(x,\bar{x})\, d\mu(x) < \infty$ for some (and thus any) $\bar{x}$

plays an important role in the proof of many estimates (e.g. contractivity properties) involving W_2.

The distance W_2 induces on $\mathscr{P}_2(X)$ the topology of weak convergence with quadratic moments, i.e. continuity of all the integrals $\mu \mapsto \int_X \phi \, d\mu$ with $\phi : X \to \mathbb{R}$ continuous and with at most quadratic growth. The metric space $(\mathscr{P}_2(X), W_2)$ is complete and separable and it inherits other useful properties from (X, d) such as compactness, completeness, existence of geodesics, nonnegative sectional curvature (see e.g. [7, 93, 104]). Particularly relevant for our discussion are the geodesic properties. In the same spirit of the concepts illustrated in Section 3 (regular Lagrangian flows, test plans, superposition principle, etc.) there is a close connection between $\mathrm{Geo}(X)$ and $\mathrm{Geo}(\mathscr{P}_2(X))$: very informally we can say that "a geodesic in the space of random variables is always induced by a random variable in the space of geodesics".

Proposition 4.1. *Any* $\boldsymbol{\eta} \in \mathscr{P}(\mathrm{Geo}(X))$ *with* $(e_0, e_1)_\# \boldsymbol{\eta}$ *optimal transport plan induces* $\mu_t := (e_t)_\# \boldsymbol{\eta} \in \mathrm{Geo}(\mathscr{P}_2(X))$*. Conversely, any* $\mu_t \in \mathrm{Geo}(\mathscr{P}_2(X))$ *is representable (in general not uniquely) in this way.*

In the sequel we shall denote by $\mathrm{OptGeo}(X)$ the *optimal geodesic plans*, so that, after recognizing that gradient velocity fields $\boldsymbol{b}_t = \nabla\phi_t$ are the optimal ones in (BB) (see also (4.6) below), we may interpret the (BB) formula by saying that W_2 is the Riemannian distance associated to the metric g (see also [73], with calculations of curvature tensors in $\mathscr{P}_2(X)$ along these lines). Similarly, according to this calculus, the heat equation can be interpreted as the gradient flow with respect to this "Riemannian structure"; as illustrated in [86, 85] and many subsequent papers (see e.g. [32] and other references in [7]), this provides a very powerful heuristic principle which applies to many more PDE's (Fokker-Planck equation, porous medium equation, etc.) and to the proof of functional/geometric inequalities. Particularly relevant for the subsequent developments is the formula

$$\int_{\{\varrho>0\}} \frac{|\nabla\varrho|^2}{\varrho}\, d\mathfrak{m} = |\nabla^-\mathrm{Ent}|^2(\varrho\mathfrak{m}) \tag{4.5}$$

which corresponds to the energy dissipation rate of Ent along the heat equation, when seen from the classical, "Eulerian", point of view (the left hand side) and from the new, "Lagrangian", point of view (the right hand side). The left hand side, also called *Fisher information*, can be written in the form $4\int|\nabla\sqrt{\varrho}|^2\, d\mathfrak{m}$.

After these discoveries, many attempts have been made to develop a systematic theory based on Otto's calculus, even though no approach based on local coordinates seems to be possible. In this direction, the building block in [7] is the identification of absolutely continuous curves in $(\mathscr{P}_2(X), W_2)$ (a purely metric notion) with solutions to the continuity equation (a notion that appeals also to the differentiable structure).

Theorem 4.2. *Assume that either $X = \mathbb{R}^N$, or X is a compact Riemannian manifold. Then, for any $\mu_t \in \mathrm{AC}^2([0,1];(\mathscr{P}_2(X), W_2))$ there exists a velocity field $\boldsymbol{b}_t$ such that the continuity equation $\frac{d}{dt}\mu_t + \mathrm{div}\,(\boldsymbol{b}_t\mu_t) = 0$ holds, and $\int_X |\boldsymbol{b}_t|^2\, d\mu_t \leq |\mu_t'|^2$ for $\mathscr{L}^1$-a.e. $t \in (0,1)$. Conversely, for any solution $(\mu_t, \tilde{\boldsymbol{b}}_t)$ to the continuity equation with $\int_0^1\int_X |\tilde{\boldsymbol{b}}_t|^2\, d\mu_t dt < \infty$ one has that $\mu_t \in \mathrm{AC}^2([0,1];(\mathscr{P}_2(X), W_2))$ with $|\mu_t'|^2 \leq \int_X |\tilde{\boldsymbol{b}}_t|^2\, d\mu_t$ for $\mathscr{L}^1$-a.e. $t \in (0,1)$. Finally, the minimal velocity field $\boldsymbol{b}_t$ is characterized by*

$$\boldsymbol{b}_t \in \overline{\{\nabla\phi:\ \phi \in C^\infty_c(X)\}}^{L^2(\mu_t)} \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0,1). \tag{4.6}$$

While for applications to PDE's it is very useful to transfer differential information from X to $\mathscr{P}_2(X)$, it has been realized only more recently that also the converse path can be useful, namely we may try to use information at the level of $\mathscr{P}_2(X)$ to get information on the energy/differentiable structure of X, or its curvature, that seem to be difficult to obtain, or to state, with different means. Besides the Lott-Villani and Sturm theory, one of the first applications of this viewpoint and of the identification (4.5) has been the following result from [17] (the full strength of the analogous identification (5.15) in $\mathrm{CD}(K,\infty)$ spaces will also play an important role in Section 6).

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Theorem 4.3. *Let X be an Hilbert space and let $\mathfrak{m} \in \mathscr{P}(X)$ be log-concave, i.e.*

$$\log \mathfrak{m}((1-t)A + tB) \geq (1-t)\log \mathfrak{m}(A) + t\log \mathfrak{m}(B) \qquad \forall t \in [0,1]$$

for any pair of open sets A, B in X. Then the quadratic form

$$\mathcal{E}(f) := \int_X |\nabla f|^2 \, d\mathfrak{m} \qquad f \text{ smooth, cylindrical}$$

is closable in $L^2(X,\gamma)$, and its closure is a Dirichlet form.

While traditional proofs of closability use quasi-invariant directions (whose existence is an open problem for general log-concave measures), here the proof is based on (4.5): lower semicontinuity of $|\nabla^- \mathrm{Ent}|$, granted by the convexity of Ent along W_2-geodesics, provides lower semicontinuity in $L^1_+(X,\mathfrak{m})$ of Fisher information, and then closability of $\mathcal{E}$.

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5 Curvature-dimension conditions

In this section I will illustrate two successful theories dealing with synthetic notions of Ricci bounds from below and dimension bounds from above. The first one, the Bakry-Émery theory, can be formulated at different levels of smoothness; I have chosen to describe it at the level of Dirichlet forms and Γ-calculus (see Section 3.2), since at this level the comparison with the Lott-Villani and Sturm theory (or, better, the Riemannian part of it) is by now well understood.

In the Bakry-Émery theory the starting point is Bochner-Lichnerowicz's formula

$$\frac{1}{2}\Delta_{\mathrm{g}}(|\nabla f|^2) - \langle \nabla f, \nabla \Delta_{\mathrm{g}} f\rangle = |\mathrm{Hess}\, f|^2 + \mathrm{Ric}(\nabla f, \nabla f), \tag{5.1}$$

valid in Riemannian manifolds, and its modification, accounting for the weight, on weighted Riemannian manifolds. We have already seen in (3.2) that the natural operator in the weighted setting is $\Delta f = \Delta_{\mathrm{g}} f - \langle \nabla V, \nabla f\rangle$, and the replacement of Δ_{g} with Δ in the left hand side of (5.1) gives

$$\frac{1}{2}\Delta(|\nabla f|^2) - \langle \nabla f, \nabla \Delta f\rangle = |\mathrm{Hess}\, f|^2 + \mathrm{Ric}_{\mathfrak{m}}(\nabla f, \nabla f) \tag{5.2}$$

where now $\mathrm{Ric}_{\mathfrak{m}}$ is the "weighted" Ricci tensor

$$\mathrm{Ric}_{\mathfrak{m}} := \mathrm{Ric} + \mathrm{Hess}\, V.$$

Still in the smooth setting, the starting point of the CD theory, instead, is a concavity inequality satisfied by the Jacobian function

$$\mathcal{J}(s,x) := \det\left[\nabla_x \exp(s\nabla\phi(x))\right],$$

in N-dimensional manifolds with $\mathrm{Ric} \geq K\,\mathrm{g}$, as long as $\mathcal{J}(s,x) > -\infty$. Namely, a careful ODE analysis (see [104, Thm. 14.12]) shows that

$$\mathcal{J}^{1/N}(s,x) \geq \tau_{K,N}^{(s)}(\theta)\mathcal{J}^{1/N}(1,x) + \tau_{K,N}^{(1-s)}(\theta)\mathcal{J}^{1/N}(0,x) \qquad \forall s \in [0,1] \quad (5.3)$$

where $\theta = \mathsf{d}(x, \exp(\nabla\phi(x)))$, $\tau_{K,N}^{(s)}(\theta) = s^{1/N}\sigma_{K/(N-1)}^{(s)}(\theta)^{1-1/N}$ and, for $s \in [0,1]$,

$$\sigma_\kappa^{(s)}(\theta) := \begin{cases} \frac{\mathfrak{s}_\kappa(s\theta)}{\mathfrak{s}_\kappa(\theta)} & \text{if } \kappa\theta^2 \neq 0 \text{ and } \kappa\theta^2 < \pi^2, \\ s & \text{if } \kappa\theta^2 = 0, \\ +\infty & \text{if } \kappa\theta^2 \geq \pi^2, \end{cases} \quad (5.4)$$

with

$$\mathfrak{s}_\kappa(r) := \begin{cases} \frac{\sin(\sqrt{\kappa}r)}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\ r & \text{if } \kappa = 0, \\ \frac{\sinh(\sqrt{-\kappa}r)}{\sqrt{-\kappa}} & \text{if } \kappa < 0. \end{cases} \quad (5.5)$$

For $\kappa\theta < \pi^2$, the coefficients $\sigma_\kappa^{(s)}(\theta)$ solve the ODE $\sigma'' + \kappa\theta^2\sigma = 0$ on $[0,1]$, with $\sigma(0) = 0$, $\sigma(1) = 1$. In the limit as $N \to \infty$ the inequality (5.3) becomes

$$\log \mathcal{J}(s,x) \geq s \log \mathcal{J}(1,x) + (1-s) \log \mathcal{J}(0,x) + K\frac{s(1-s)}{2}\theta^2. \quad (5.6)$$

5.1 The BE theory

In the framework of Dirichlet forms and Γ-calculus (see Section 3.2) there is still the possibility to write (5.2) in the weak form of an inequality. Let us start from the observation that, because of the locality assumption, one has $\Gamma(f) = \frac{1}{2}\Delta f^2 - f\Delta f$. Now, we may write (5.2) in terms of the *iterated*

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It is not hard to see that this is a *strongly consistent* definition of upper bound on dimension and lower bound on Ricci tensor, in the smooth setting of weighted n-dimensional Riemannian manifolds: more precisely when V is constant $\mathrm{BE}(K,n)$ holds if and only if $\mathrm{Ric} \geq K\mathrm{g}$ and, when $N > n$, $\mathrm{BE}(K,N)$ holds if and only if

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Euclidean spaces such as the logarithmic entropy in (4.4), introducing the notion of *displacement convexity*, i.e. convexity along $\mathrm{Geo}(\mathscr{P}_2(X))$. More generally, by considering the dimensional counterparts of Ent, Rényi's entropies

$$\mathcal{E}_N(\mu) := -\int_X$$

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even in the metric setting (with a few exceptional points in space-time). From (5.14) one can obtain [70] another key connection between the Lagrangian and Eulerian points of view: the estimate of metric derivative with Fisher information:

$$|\mu_t'|^2 \le \int_{\{\mathsf{P}_t\varrho>0\}} \frac{|\nabla \mathsf{P}_t\varrho|_*^2}{\mathsf{P}_t\varrho}\,d\mathfrak{m} \quad \text{for } \mathscr{L}^1\text{-a.e. } t>0, \text{ with } \mu_t := \mathsf{P}_t\varrho\,\mathfrak{m}.$$

Theorem 5.3. *Let $(X,\mathsf{d},\mathfrak{m})$ be a $\mathrm{CD}(K,\infty)$ metric measure space and let $\varrho \in L^1(X,\mathfrak{m})$ be non-negative with $\varrho\mathfrak{m} \in \mathscr{P}_2(X)$. Then:*

(a) the curve of measures $\mu_t := \mathsf{P}_t\varrho\,\mathfrak{m}$ is the unique W_2-gradient flow of Ent *starting from $\varrho\mathfrak{m}$;*

(b) $|\nabla^-\mathrm{Ent}|(\varrho\mathfrak{m})$ is finite if and only if $\mathrm{Ch}_2(\sqrt{\varrho}) < \infty$ and

$$|\nabla^-\mathrm{Ent}|^2(\varrho\mathfrak{m}) = \int_{\{\varrho>0\}} \frac{|\nabla\varrho|_*^2}{\varrho}\,d\mathfrak{m} = 4\,\mathrm{Ch}_2(\sqrt{\varrho}). \tag{5.15}$$

Even though the two notions of heat flow can be identified, they are conceptually different and their natural domains differ (in particular when, thanks to contractivity, the W_2-gradient flow of Ent can be extended to the whole of $\mathscr{P}_2(X)$). For this reason we will use the distinguished notation

$$\mathscr{H}_t\mu := \mathsf{P}_t\varrho\,\mathfrak{m} \qquad \text{whenever } \mu = \varrho\mathfrak{m} \in \mathscr{P}_2(X). \tag{5.16}$$

Let us move now to the "dimensional" theory, i.e. when we want to give an upper bound $N < \infty$ on the dimension, with $N > 1$. In this case the convexity conditions should take into account also the parameter N, and the distorsion coefficients $\tau_{K,N}^{(s)}(\theta) = s^{1/N}\sigma_{K/(N-1)}^{(s)}(\theta)^{1-1/N}$ are those of (5.3), (5.4). In this text I follow more closely Sturm's axiomatization [102, 103] (J.Lott and C.Villani's one [74] uses a more general classes of entropies, not necessarily power-like, singled out by R.McCann).

Definition 5.4 ($\mathrm{CD}(K,N)$ spaces)**.** We say that $(X,\mathsf{d},\mathfrak{m})$ satisfies the curvature dimension condition $\mathrm{CD}(K,N)$ if the functionals $\mathcal{E}_M$ in (5.12) satisfy: for all $\mu_0 = \varrho_0\mathfrak{m}$, $\mu_1 = \varrho_1\mathfrak{m} \in \mathscr{P}_2(X)$ with bounded support there exists $\boldsymbol{\eta} \in \mathrm{OptGeo}(\mu_0,\mu_1)$ with

$$\mathcal{E}_{N'}(\mu_s) \le -\int [\tau_{K,N'}^{(1-s)}(\mathsf{d}(\gamma_0,\gamma_1))\varrho_0^{-1/N'}(\gamma_0) + \tau_{K,N'}^{(s)}(\mathsf{d}(\gamma_0,\gamma_1))\varrho_1^{-1/N'}(\gamma_1)]\,d\boldsymbol{\eta}(\gamma) \tag{5.17}$$

for all $N' \ge N$ and $s \in [0,1]$, where $\mu_s := (e_s)_\sharp\boldsymbol{\eta}$.

Besides N-dimensional Riemannian manifolds with Ricci $\ge K\,\mathrm{Id}$ and Finsler manifolds [83], it has been proved by A.Petrunin in [90] that the class $\mathrm{CD}(0,N)$ includes also positively curved N-dimensional spaces, in the

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sense of Alexandrov. The definition is built in such a way that the curvature dimension condition becomes weaker as N increases, and it implies (by taking $N' \to \infty$ in (5.17) and using that $N' + N'\mathcal{E}_{N'} \to \mathrm{Ent}$) the $\mathrm{CD}(K,\infty)$ condition. These curvature dimension conditions, besides being stable w.r.t. m-GH convergence, can be used to establish functional and geometric inequalities, often with sharp constants, see Section 5.3. However, except in the cases $K = 0$ or $N = \infty$ (and under the non-branching assumption) it is not clear why the CD condition holds globally, when it holds locally, and T.Rajala built indeed in [91] a highly branching $\mathrm{CD}_{\mathrm{loc}}(0,4)$ space which, for no value of K and N, is $\mathrm{CD}(K,N)$. This *globalization* problem is a fundamental issue, since only the global condition, without artificial scale factors, can be proved to be stable w.r.t. m-GH convergence. Recently, in the class of essentially non-branching m.m.s. (see Definition 5.6 below), the globalization problem has been brilliantly solved by F.Cavalletti and E.Milman in [33], building on a very refined analysis of the metric Hamilton-Jacobi equation (5.14) and the regularity of Q_t. The globalization problem led K.Bacher and K.T.Sturm to the introduction in [19] of a weaker curvature-dimension condition CD^*, involving the smaller coefficients $\sigma_\kappa^{(s)}(\theta)$:

Definition 5.5 ($\mathrm{CD}^*(K,N)$ spaces). We say that $(X,\mathsf{d},\mathfrak{m})$ satisfies the reduced curvature dimension condition $\mathrm{CD}^*(K,N)$ if the functionals $\mathcal{E}_M$ in (5.12) satisfy: for all $\mu_0 = \varrho_0\mathfrak{m}$, $\mu_1 = \varrho_1\mathfrak{m} \in \mathscr{P}_2(X)$ with bounded support there exists $\boldsymbol{\eta} \in \mathrm{OptGeo}(\mu_0,\mu_1)$ with

$$\mathcal{E}_{N'}(\mu_s) \le -\int [\sigma_{K/N'}^{(1-s)}(\mathsf{d}(\gamma_0,\gamma_1))\varrho_0^{-1/N'}(\gamma_0)+\sigma_{K/N'}^{(s)}(\mathsf{d}(\gamma_0,\gamma_1))\varrho_1^{-1/N'}(\gamma_1)]\, d\boldsymbol{\eta}(\gamma) \tag{5.18}$$

for all $N' \ge N$ and $s \in [0,1]$, where $\mu_s := (e_s)_\sharp\boldsymbol{\eta}$.

At the local level the two classes of spaces coincide, more precisely

$$\bigcap_{K'<K} \mathrm{CD}^*_{\mathrm{loc}}(K',N) \sim \bigcap_{K'<K} \mathrm{CD}_{\mathrm{loc}}(K',N).$$

In addition, the inclusion $\mathrm{CD}(K,N) \subset \mathrm{CD}^*(K,N)$ can be reversed at the price of replacing, in $\mathrm{CD}(K,N)$, K with $K^* = K(N-1)/N$ (in particular one can still obtain from $\mathrm{CD}^*(K,N)$ functional inequalities, but sometimes with non-optimal constants). More results can be established in the class of the *essentially non-branching* m.m.s., first singled out in [92]. Recall that a metric space (X,d) is said to be non-branching if the map $(e_0,e_t) : \mathrm{Geo}(X) \to X^2$ is injective for all $t \in (0,1]$ (for instance Riemannian manifolds and Alexandrov spaces are non-branching). Analogously we can define the non-branching property of a subset E of $\mathrm{Geo}(X)$.

Definition 5.6 (Essential non-branching). We say that $(X,\mathsf{d},\mathfrak{m})$ is essentially non-branching if any $\boldsymbol{\eta} \in \mathrm{OptGeo}(\mu_0,\mu_1)$ with $\mu_i \in \mathscr{P}_2(X)$ and $\mu_i \ll \mathfrak{m}$ is concentrated on a Borel set of non-branching geodesics.

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It has been proved in [92] that *strong* $\mathrm{CD}(K,\infty)$ spaces are essentially non-branching. In the class of essentially non-branching m.m.s. the CD^* condition gains the local-to-global property, namely $\mathrm{CD}^*_{\rm loc}(K,N)\sim\mathrm{CD}^*(K,N)$.

Finally, we can complete the list of CD spaces with the *entropic* CD^e spaces, introduced in [49]. Their definition involves the new notion of (K,N)-*convexity*. In a geodesic space X, a function S is said to be (K,N)-convex if for any pair of points $\gamma_0,\,\gamma_1\in X$ there exists $\gamma\in\mathrm{Geo}(X)$ connecting these two points such that $(S\circ\gamma)''\geq K\mathsf{d}^2(\gamma_1,\gamma_0)+|(S\circ\gamma)'|^2/N$ in $(0,1)$, in the sense of distributions. In the smooth setting, this is equivalent to either the inequalities

$$\mathrm{Hess}\,S\geq K\,\mathrm{Id}+\frac{1}{N}(\nabla S\otimes\nabla S),\qquad \mathrm{Hess}\,S_N\leq-\frac{K}{N}S_N \tag{5.19}$$

for $S_N:=\exp(-S/N)$, while in the metric setting this property can be formulated in terms of the inequality

$$S_N(\gamma_t)\geq\sigma^{(1-t)}_{K/N}(\mathsf{d}(\gamma_0,\gamma_1))S_N(\gamma_0)+\sigma^{(t)}_{K/N}(\mathsf{d}(\gamma_0,\gamma_1))S_N(\gamma_1)\qquad t\in[0,1],,\ \gamma\in\mathrm{Geo}(X).$$

These facts, and the differential inequality $\ell''(s)\geq(\ell'(s))^2/n+\mathrm{Ric}\,(\gamma'(s),\gamma'(s))$, valid in the smooth setting with $\ell(s)=-\log\mathcal{J}(s,x)$ and $\gamma(s)=\exp(s\nabla\phi(x))$, motivate the following definition.

Definition 5.7 ($\mathrm{CD}^e(K,N)$ spaces)**.** We say that $(X,\mathsf{d},\mathfrak{m})$ satisfies the entropic curvature dimension condition $\mathrm{CD}^e(K,N)$ if the functional Ent is (K,N)-convex in $\mathscr{P}_2(X)$.

The following result (due for the first part to [49], for the second part to [33]) provides, under the essential non-branching assumption, a basic equivalence between all these definitions. In addition, [33] provides also equivalence with another definition based on disintegrations of $\mathfrak{m}$ (as in Alberti's representations mentioned in Section 3.1) induced by transport rays of the optimal transport problem with cost=distance.

Theorem 5.8 (Equivalence under essential non-branching)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an essentially non-branching m.m.s. with $\mathfrak{m}(X)<\infty$. Then $(X,\mathsf{d},\mathfrak{m})$ is $\mathrm{CD}^e(K,N)$ iff it is $\mathrm{CD}^*(K,N)$ iff it is $\mathrm{CD}(K,N)$.*

Finally, inspired by the calculations done in the smooth setting in [104] (see (29.2) therein) we also proved in [15] that, for essentially non-branching m.m.s., the $\mathrm{CD}^*(K,N)$ condition is equivalent to a distorted convexity inequality for Rényi's entropy

$$\mathcal{E}_N(\mu_s)\leq(1-s)\mathcal{E}_N(\mu_0)+s\mathcal{E}_N(\mu_1)-K\mathcal{A}^{(s)}_N(\mu)\qquad\forall s\in[0,1],$$

where the dimensional distorsion is present also in the action term

$$\mathcal{A}^{(s)}_N(\mu):=\int_0^1\int_X G(t,s)\varrho_t^{1-1/N}|v_t|^2\,d\mathfrak{m}\,dt\qquad \mu_t=\varrho_t\mathfrak{m}+\mu_t^\perp.$$

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Here $|v_s|$ is the minimal velocity field of μ_s (which still makes sense in the metric setting, by an adaptation of Theorem 4.2) and G is a suitable Green function. In the limit as $N \to \infty$, along geodesics $\mu_s \ll \mathfrak{m}$, the action term converges to $\frac{1}{2}s(1-s)\mathrm{W}_2^2(\mu_0, \mu_1)$.

5.3 Geometric and functional inequalities

We recall here some of the most important geometric and functional inequalities by now available in the settting of $\mathrm{CD}(K, N)$ spaces.

Bishop-Gromov inequality and Bonnet-Myers diameter estimate: [104] The map

$$r \mapsto \frac{\mathfrak{m}(B_r(x_0))}{\int_0^r s^{K,N}(t)\,\mathrm{d}t} \quad \text{is nonincreasing for all } x_0 \in X.$$

When N is an integer $s^{K,N}$ can be interpreted as the functions providing the measure of the spheres in the model space of Ricci curvature K and dimension N. If $K > 0$ the diameter of X is bounded by $\pi\sqrt{(N-1)/K}$.

Upper bounds on $\Delta \mathsf{d}^2$: [?] Under a suitable strict convexity assumption of Ch_2, in $\mathrm{CD}^*(K, N)$ spaces one has the upper bound $\Delta \mathsf{d}^2 \leq \gamma_{K,N}(\mathsf{d})\mathfrak{m}$ in the weak sense (with $\gamma_{0,N} \equiv 2N$).

Spectral gap and Poincaré inequality: If $K > 0$ then

$$\int_X (f - \bar{f})^2 \, d\mathfrak{m} \leq \frac{N-1}{NK} \int_X |\nabla f|_*^2 \, d\mathfrak{m}, \quad \text{with} \quad \bar{f} = \int_X f \, d\mathfrak{m}.$$

In more recent times, B.Klartag used L^1 optimal transportation methods and the *localization* technique (going back to the work of Payne-Weinberger [89] and then further developed in the context of convex geometry by Gromov-Milman and Kannan-Lovàsz- Simonovitz) to provide in [67] a new proof of the Levy-Gromov isoperimetric inequality in Riemannian manifolds, one of the few inequalities not available with Γ-calculus tools. Shortly aftwerwards, F.Cavalletti and A.Mondino have been able to extend in [34, 35] the localization technique to obtain in the class of essentially non-branching $\mathrm{CD}(K, N)$ m.m.s. this and many other inequalities with sharp constants.

Levy-Gromov inequality: [34] If $\mathfrak{m}(X) = 1$ and $K > 0$, then for any Borel set $E \subset X$ one has

$$\mathfrak{m}^+(E) \geq \frac{|\partial B|}{|S|}$$

where $\mathfrak{m}^+(E) = \liminf_{r\downarrow 0}(\mathfrak{m}(E_r) - \mathfrak{m}(E))/r$ is the Minkowski content of E (coinciding with the perimeter of the boundary, for sufficiently nice sets E) and B is a spherical cap in the N-dimensional sphere S with Ricci curvature equal to K such that $|B|/|S| = \mathfrak{m}(E)$. This is part of a more general isoperimetric statement proved in [34] involving isoperimetric profiles and model spaces for manifolds with dimension smaller than N, Ricci curvature larger

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than K and diameter smaller than D discovered in [78]. In $\mathrm{RCD}(K,\infty)$ spaces see also [23, Cor. 8.5.5], [14].

Log-Sobolev and Talagrand inequalities: If $K>0$ and $\mathfrak{m}(X)=1$ then

$$\frac{KN}{2(N-1)}\mathrm{W}_2^2(\varrho\mathfrak{m},\mathfrak{m})\le \mathrm{Ent}(\varrho\mathfrak{m})\le \frac{N-1}{2KN}\int_{\{\varrho>0\}}\frac{|\nabla\varrho|_*^2}{\varrho}\,d\mathfrak{m}.$$

Sobolev inequalities: If $K>0$, $N>2$, $2<p\le 2N/(N-2)$, then (see also [23, Thm. 6.8.3])

$$\|f\|_{L^p}^2\le \|f\|_{L^2}^2+\frac{(p-2)(N-1)}{KN}\int_X|\nabla f|_*^2\,d\mathfrak{m}.$$

6 Stability of curvature-dimension bounds and heat flows

In this section we deal with *pointed* m.m.s. $(X,\mathsf{d},\mathfrak{m},\bar x)$, a concept particularly useful when (X,d) has infinite diameter and blow-up procedures are performed. Pointed metric measure structures are identified by measure-preserving isometries of the supports which preserve the base points. Remarkably, Gromov's *reconstruction theorem* [60] (extended in [56] to spaces with infinite mass), characterizes the equivalence classes by the family of functionals

$$\varphi^\star\big[(X,\mathsf{d},\mathfrak{m},\bar x)\big]:=\int_{X^N}\varphi\big(\mathsf{d}(x_i,x_j)_{i,j=1}^N\big)\,d\delta_{\bar x}(x_1)\,d\mathfrak{m}^{\otimes N-1}(x_2,\dots,x_N), \tag{6.1}$$

where $N\ge 2$ and $\varphi:\mathbb{R}^{N^2}\to\mathbb{R}$ is continuous with bounded support.

A fundamental property of the CD condition is the stability w.r.t. (pointed) measured Gromov-Hausdorff convergence, established (in slightly different settings) in [74, 102, 103]. Building on Gromov's seminal work [60] on convergence for metric structures, this notion of convergence for (pointed) metric measure structures was introduced by K.Fukaya in connection with spectral stability properties, and then it has been a crucial ingredient in the remarkable program developed in the 90's by J.Cheeger and T.Colding [37, 38, 39, 40], dealing with the fine structure of Ricci limit spaces (particularly in the collapsed case).

According to local and global assumptions on the sequence of metric measure structures, several definitions of convergence are possible. For the sake of illustration, I follow here the definition of *pointed measured Gromov convergence* in [56, 58], based on the reconstruction theorem. As for Sturm's $\mathbb{D}$-convergence [102], this notion of convergence, while avoiding at the same time finiteness of the measure and local compactness, is consistent with pointed mGH-convergence when the pointed m.m.s. have more structure (e.g. under a uniform doubling condition, ensured in the $\mathrm{CD}(K,N)$ case,

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$N < \infty$, by the Bishop-Gromov inequality). Within this approach, not relying on doubling and local compactness, general $\mathrm{CD}(K,\infty)$ spaces can also be treated (see also [102, 98] for a comparison with Gromov's notions [60] of box and concentration convergence).

Definition 6.1 (pmG-convergence). We say that $(X^h, \mathsf{d}^h, \mathfrak{m}^h, \bar{x}^h)$ converge to $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ if for every functional $\varphi^\star$ as in (6.1) one has

$$\lim_{h\to\infty} \varphi^\star\big[(X^h, \mathsf{d}^h, \mathfrak{m}^h, \bar{x}^h)\big] = \varphi^\star\big[(X, \mathsf{d}, \mathfrak{m}, \bar{x})\big].$$

The following result from [56], which includes as a particular case those proved in [40] for Ricci limit spaces and those proved in [97] for Finsler manifolds, provides not only stability of the $\mathrm{CD}(K,\infty)$ condition, but also convergence of Cheeger's energies and heat flows; for Cheeger's energies, the right notion of convergence is Mosco convergence [81], a notion of variational convergence particularly useful in connection to stability of variational inequalities, that can be adapted also to the case when sequences of metric measure structures are considered.

Theorem 6.2. *Assume that* $(X^h, \mathsf{d}^h, \mathfrak{m}^h, \bar{x}^h)$ *are* $\mathrm{CD}(K,\infty)$ *pointed m.m.s., pmG-convergent to* $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$. *Then:*

(a) $(X, \mathsf{d}, \mathfrak{m})$ *is* $\mathrm{CD}(K,\infty)$*;*

(b) the Cheeger energies $\mathrm{Ch}_{2,h}$ *relative to* $(X^h, \mathsf{d}^h, \mathfrak{m}^h)$ *Mosco converge to the Cheeger energy* Ch_2 *relative to* $(X, \mathsf{d}, \mathfrak{m})$*;*

(c) the heat flows P_t^h *relative to* $(X^h, \mathsf{d}^h, \mathfrak{m}^h)$ *converge to the heat flow* P_t *relative to* $(X, \mathsf{d}, \mathfrak{m})$.

In order to give a mathematically rigorous and specific meaning to (b) and (c) one has to use the so-called *extrinsic* approach, embedding isometrically all spaces into a single complete and separable metric space (Z, d_Z); within this realization of the convergence, which is always possible, pmG-convergence corresponds to weak convergence of $\mathfrak{m}^h$ to $\mathfrak{m}$. The proof of parts (b) and (c) of Theorem 6.2 relies once more on Theorem 5.3 and particularly on the key identification (5.15), to transfer information from the Lagrangian level (the one encoded in the definition of convergence) to the Eulerian level.

7 Adding the Riemannian assumption

One of the advantages of the CD theory, when compared to the BE theory dealing essential with quadratic energy structures, is its generality: it provides a synthetic language to state and prove functional and geometric inequalities in structures which are far, even on small scales, from being Euclidean. On the other hand, as advocated in [59] and [38, Appendix 2], the

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description of the closure with respect to m-GH convergence of Riemannian manifolds requires a finer axiomatization, possibly based on the linearity of the heat flow. Within the CD theory, a good step forward in this direction has been achieved in [10], see also [6]:

Definition 7.1 (RCD(K,∞) condition). A $(X,\mathsf{d},\mathfrak{m})$ m.m.s. satisfies the RCD(K,∞) condition if it is CD(K,∞) and Ch_2 is a quadratic form, i.e. if $(X,\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian according to Definition 3.8.

This new definition is useful (for instance in the proof of rigidity results by compactness arguments) only if the additional "Riemannian" axiom, equivalent to the linearity of the semigroup P_t, is stable with respect to the measured Gromov-Hausdorff convergence and its variants. Simple examples show that, by itself, it is not. However, the remarkable fact is that the extra axiom is stable, when combined with the CD(K,∞) condition. This stability property could be seen as a consequence of Mosco convergence (see Theorem 6.2), since quadraticity is stable under Mosco convergence. However, the original proof of stability of the RCD(K,∞) condition given in [10] uses the full strength of the Riemannian assumption and relies on the characterization of RCD(K,∞) in terms of the EVI_K-property of the heat flow $\mathscr{H}_t$.

Theorem 7.2 ([10], [6]). *$(X,\mathsf{d},\mathfrak{m})$ is* RCD(K,∞) *if and only if the heat semigroup $\mathscr{H}_t$ in (5.16) satisfies the* EVI_K *property*

$$\frac{d}{dt}\frac{1}{2}W_2^2(\mathscr{H}_t\mu,\nu) \le \mathrm{Ent}(\nu) - \mathrm{Ent}(\mathscr{H}_t\mu) - \frac{K}{2}W_2^2(\mathscr{H}_t\mu,\nu) \tag{7.1}$$

for all initial datum $\mu = \varrho\mathfrak{m} \in \mathscr{P}_2(X)$, and all $\nu \in \mathscr{P}_2(X)$.

Since EVI_K solutions are metric gradient flows, the previous theorem could also have been stated in terms of a semigroup satisfying the EVI_K property (this formulation, only apparently weaker, is useful for instance in connection with the stability of heat flows in the RCD setting). EVI_K solutions are a crucial technical tool for more than one reason: first, as we have seen in Theorem 7.2, they encode in a single condition both the CD and the Riemannian assumption; even more (see [7, 45] for a more complete account of the EVI theory), they enjoy strong stability and contractivity properties that allow at once the extension of $\mathscr{H}_t$ to the whole of $\mathscr{P}_2(X)$, with $W_2(\mathscr{H}_t\mu,\mathscr{H}_t\nu) \le e^{-Kt}W_2(\mu,\nu)$. Finally, S.Daneri and G.Savaré discovered in [44] that the existence of EVI_K solutions, for a given function S in a geodesic space, encodes also the strong convexity (i.e. convexity along all constant speed geodesics). As a consequence, RCD(K,∞) spaces are strong CD(K,∞) spaces and we obtain from [92] also the essential non-branching property of this new class of spaces.

In [10] we proved several properties of RCD spaces, and many more have been proved in subsequent papers (see the next section). To conclude

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this section, I will describe results which establish an essential equivalence between the RCD and the BE theories, both in the dimensional and adimensional case. The connection can be established in one direction using Cheeger's energy Ch_2, in the other direction using the intrinsic distance $\mathsf{d}_{\mathcal{E}}$. A precursor of these results is K.Kuwada's paper [70], which first provided the equivalence in the Riemannian setting of gradient contractivity $|\nabla \mathsf{P}_t f|^2 \leq e^{-2Kt}\mathsf{P}_t|\nabla f|^2$ (namely the integrated form of $\mathrm{BE}(K,\infty)$) and contractivity of W_2 under the heat flow. The advantages of this "unification" of the theories are evident: at the RCD level one can use (with the few limitations I already mentioned) all power of Γ-calculus, having at the same time all stability and geometric properties granted by the metric point of view.

For the sake of simplicity, I will state the next results for the case when $\mathfrak{m}$ is finite measure, but most results have been proved also in the more general setting, under suitable global assumptions analogous to (5.11).

Theorem 7.3 ([10], [

illustrated in [23] (and then used also in [5], in the class of *extended* m.m.s.) involves instead the dual representation (4.2) of W_2^2.

Moving now to the dimensional case, the following definition (first proposed in [54]) is natural.

Definition 7.4 ($\mathrm{RCD}^*(K,N)$ condition). For $N \ge 1$, a $\mathrm{CD}^*(K,N)$ m.m.s. $(X,\mathsf{d},\mathfrak{m})$ satisfies the $\mathrm{RCD}^*(K,N)$ condition if Ch_2 is a quadratic form, i.e. if $(X,\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian according to Definition 3.8.

In light of the recent equivalence result [33] between the CD^* and CD conditions in essential non-branching m.m.s. (since, as we have seen, $\mathrm{RCD}(K,\infty)$ spaces are essentially non-branching), we now know that $\mathrm{RCD}^*(K,N)$ is equivalent to $\mathrm{RCD}(K,N)$, i.e. $\mathrm{CD}(K,N)$ plus infinitesimally Hilbertian.

Building on Theorem 7.2, the equivalence between the $\mathrm{BE}(K,N)$ and $\mathrm{RCD}^*(K,N)$ with $N<\infty$ has been proved, independently, in [49] and [15]. The "distorsion" of the EVI_K property due to the dimension has been treated quite differently in the two papers: in [49], instead of Rényi's entropies, a suitable dimensional modification of Ent, the so-called power entropy functional

$$\mathrm{Ent}_N(\mu) := \exp\left(-\frac{1}{N}\mathrm{Ent}(\mu)\right) \tag{7.3}$$

has been used. We have already seen in (5.19) that, in the smooth setting, the (K,N)-convexity condition for S can also be reformulated in terms of $S_N = \exp(-S/N)$. It turns out that, still in a Riemannian setting, the (K,N)-convexity condition can be formulated in terms of a $\mathrm{EVI}_{K,N}$ condition satisfied by the gradient flow γ_t of S: more precisely

$$\frac{d}{dt}\mathfrak{s}^2_{K/N}\left(\frac{1}{2}\mathsf{d}(\gamma_t,z)\right) + K\mathfrak{s}^2_{K/N}\left(\frac{1}{2}\mathsf{d}(\gamma_t,z)\right) \le \frac{N}{2}\left(1-\frac{S_N(z)}{S_N(\gamma_t)}\right)$$

for all $z\in X$, where $\mathfrak{s}_\kappa$ are defined in (5.5).

These facts are at the basis of the following result from [49].

Theorem 7.5. *$(X,\mathsf{d},\mathfrak{m})$ is a $\mathrm{RCD}^*(K,N)$ m.m.s. if and only if (X,d) is a length space and the heat semigroup $\mathscr{H}_t$ starting from any $\mu\in\mathscr{P}_2(X)$ satisfies the $\mathrm{EVI}_{K,N}$ property:*

$$\frac{d}{dt}\mathfrak{s}^2_{K/N}\left(\frac{1}{2}W_2(\mathscr{H}_t\mu,\nu)\right) + K\mathfrak{s}^2_{K/N}\left(\frac{1}{2}W_2(\mathscr{H}_t\mu,\nu)\right) \le \frac{N}{2}\left(1-\frac{\mathrm{Ent}_N(\nu)}{\mathrm{Ent}_N(\mathscr{H}_t\mu)}\right) \tag{7.4}$$

for all $\nu\in\mathscr{P}_2(X)$.

The characterization of $\mathrm{RCD}^*(K,N)$ provided in [15], involves, instead, a distorted EVI property of McCann's N-displacement convex entropies $\int_X U(\varrho)\,d\mathfrak{m}$ and their gradient flow, which is a *nonlinear diffusion equation*

$$\frac{d}{dt}\varrho_t = \Delta P(\varrho_t) \qquad \text{with} \qquad P(z) := zU'(z) - U(z).$$

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This is very much in the spirit of Otto's seminal paper [86], motivated precisely by the long term behaviour, in Euclidean spaces, of solutions to these equations.

As we will see in Section 8, distorted Evolution Variational Inequalities lead also to new contractivity estimates, besides those which already characterize the curvature-dimension condition [106] and those that can be obtained by adapting Γ-calculus techniques to the RCD setting.

8 Properties of RCD spaces

Heat kernel and contractivity. In $\mathrm{RCD}(K,\infty)$ spaces, the EVI_K-property of the heat flow immediately leads to $W_2^2(\mathscr{H}_t\mu,\mathscr{H}_t\nu)\le e^{-2Kt}W_2^2(\mu,\nu)$ and then, by duality to the gradient contractivity property $|\nabla \mathsf{P}_t f|_*^2\le e^{-2Kt}\mathsf{P}_t|\nabla f|_*^2$ and to the Feller property, namely $\mathsf{P}_t: L^\infty(X,\mathfrak{m})\to C_b(X)$, $t>0$. Wang's log-Harnack inequality [106] also implies the regularization of $\mathscr{H}_t$, $t>0$, from $\mathscr{P}_2(X)$ to absolutely continuous probability measures with density in LlogL. These inequalities can be improved, taking the dimension into account, in various ways, see [106] and the more recent papers [28, 49]. On the Lagrangian side, from (7.4) one obtains

$$\mathfrak{s}^2_{K/N}\Big(\frac12 W_2(\mathscr{H}_t\mu,\mathscr{H}_s\nu)\Big)\le e^{-K(s+t)}\mathfrak{s}^2_{K/N}\Big(\frac12 W_2(\mathscr{H}_t\mu,\mathscr{H}_s\nu)\Big)+\frac{N}{K}(1-e^{-K(s+t)})\frac{(\sqrt{s}-\sqrt{t})^2}{2(s+t)},$$

while on the Eulerian side one can recover in the RCD setting the inequality

$$|\nabla \mathsf{P}_t f|_*^2+\frac{4Kt^2}{N(e^{2Kt}-1)}|\Delta \mathsf{P}_t f|^2\le e^{-2Kt}\mathsf{P}_t|\nabla f|_*^2\qquad \mathfrak{m}\text{-a.e. on } X$$

proved by Γ-calculus techniques in [24]. In connection with nearly optimal heat kernel bounds, see [62].

Li-Yau and Harnack inequalities: If $K\ge 0$, $N<\infty$, $f>0$ then the Γ-techniques (see for instance [23, Cor. 6.7.6]) have been adapted in [52] to the RCD setting to obtain the Li-Yau and Harnack inequalities:

$$\Delta(\log \mathsf{P}_t f)\ge -\frac{N}{2t}\quad t>0,\qquad \mathsf{P}_t f(x)\le \mathsf{P}_{t+s}f(y)\left(\frac{t+s}{t}\right)^{N/2}e^{\mathsf{d}^2(x,y)/(2s)}.$$

Tensorization: Tensorization is the persistence of geometric/analytic properties when we we consider two factors $(X_1,\mathsf{d}_1,\mathfrak{m}_1)$, $(X_2,\mathsf{d}_2,\mathfrak{m}_2)$ having both these properties, and their product

$$(X_1\times X_2,\mathsf{d},\mathfrak{m}_1\times\mathfrak{m}_2)\qquad \text{with } \mathsf{d}^2((x_1',x_2'),(x_1,x_2)):=\mathsf{d}_1^2(x_1,x_1')+\mathsf{d}_2^2(x_2,x_2').$$

For instance, it is easily seen that the completeness and geodesic properties tensorize. At the level of CD spaces, we know from [103, 19, 104] that essentially non-branching $\mathrm{CD}(0,N)$, $\mathrm{CD}(K,\infty)$ and $\mathrm{CD}^*(K,N)$ spaces all have

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the tensorization property. When we add the Riemannian assumption we get the strong $\mathrm{CD}(K,\infty)$ property and then the essential non-branching property. Therefore, taking also into account the tensorization of the infinitesimally Hilbertian property [10, 16], we obtain that all spaces $\mathrm{RCD}^*(K,N)$ tensorize. Alternatively, one can use the equivalence results of Theorem 7.2 and Theorem 7.5 to obtain the tensorization from the BE theory.

Improved stability results: Thanks to the more refined calculus tools available in RCD spaces, and to the gradient contractivity available in the RCD setting, in [12] the convergence result of Theorem 6.2 has been extended to the whole class of p-th Cheeger energies Ch_p, including also the total variation norm. This gives, among other things, also the stability of isoperimetric profiles and Cheeger's constant.

Splitting theorem: In [53], N.Gigli extended to the RCD setting the Cheeger-Gromoll splitting theorem: If $K \geq 0$, $N \in [2,\infty)$ and X contains a line, i.e. there exists $\gamma : \mathbb{R} \to X$ such that $\mathsf{d}(\gamma(s),\gamma(t)) = |t-s|$ for every s, $t \in \mathbb{R}$, then $(X,\mathsf{d},\mathfrak{m})$ is isomorphic to the product of $\mathbb{R}$ and a $\mathrm{RCD}(0,N-1)$ space.

Universal cover: [80] $\mathrm{RCD}^*(K,N)$ have a universal cover, this is the first purely topological result available on this class of spaces.

Maximal diameter theorem: [66] If $(X,\mathsf{d},\mathfrak{m})$ is a $\mathrm{RCD}(N,N+1)$ space with $N>0$ and there exist $x,y \in X$ with $\mathsf{d}(x,y)=\pi$, then $(X,\mathsf{d},\mathfrak{m})$ is isomorphic to the spherical suspension of $[0,\pi]$ and a $\mathrm{RCD}(N-1,N)$ space with diameter less than π.

Volume-to-metric cones: [46] If $K=0$, there exists $\bar{x} \in X$ such that $\mathfrak{m}(B_R(\bar{x})) = (R/r)^N \mathfrak{m}(B_r(\bar{x}))$ for some $R>r>0$ and $\partial B_{R/2}(\bar{x})$ contains at least 3 points, then $B_R(\bar{x})$ is locally isometric to the ball $B_R(0)$ of the cone Y built over a $\mathrm{RCD}(N-2,N-1)$ space. This extends the Riemannian result of [37].

Local structure: The k-dimensional regular set $\mathcal{R}_k$ of a $\mathrm{RCD}^*(K,N)$-space $(X,\mathsf{d},\mathfrak{m})$ is the set of points $x \in \operatorname{supp}\mathfrak{m}$ such that

$$(X, r^{-1}\mathsf{d}, s_{x,r}\mathfrak{m}, x) \overset{m-GH}{\to} (\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, c_k\mathscr{H}^k, 0) \qquad \text{as } r \to 0^+,$$

where $c_k^{-1} = \int_{B_1(0)} (1-|x|)\, d\mathscr{H}^k(x)$, and $s_{x,r}^{-1} = \int_{B_r(x)} (1-\mathsf{d}(x,\cdot)/r)\, d\mathfrak{m}$. For $k \geq 1$ integer, we say that a set $S \subset X$ is $(\mathfrak{m},k)$-rectifiable if $\mathfrak{m}$-almost all of S can be covered by Lipschitz images of subsets of $\mathbb{R}^k$. The following theorem provides some information on the local structure of $\mathrm{RCD}^*(K,N)$ spaces, analogous to those obtained for Ricci limit spaces in [38, 39, 40]; see [79] for the proof of the first two statements (more precisely, it has been proved the stronger property that $\mathfrak{m}$-almost all of $\mathcal{R}_k$ can be covered by bi-Lipschitz charts with bi-Lipschitz constant arbitrarily close to 1) and [65, 47, 57] for the proof of the absolute continuity statement.

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Theorem 8.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a* $\mathrm{RCD}^*(K,N)$ *space with* $N \in (1,\infty)$*. For all* $k \in [1,N]$ *the set* $\mathcal{R}_k$ *is* $(\mathfrak{m},k)$*-rectifiable and*

$$\mathfrak{m}(X \setminus \bigcup_{1\le k\le N} \mathcal{R}_k) = 0.$$

In addition, the restriction $\mathfrak{m}\llcorner\mathcal{R}_k$ *of* $\mathfrak{m}$ *to* $\mathcal{R}_k$ *is absolutely continuous w.r.t.* $\mathscr{H}^k$*.*

Second order calculus: Building on Bakry's definition of Hessian (5.10), N.Gigli has been able to develop in [?] a full second-order calculus in $\mathrm{RCD}(K,\infty)$ spaces, including covariant derivatives for vector fields, connection Laplacian, Sobolev differential forms of any order and Hodge Laplacian. The starting points are, besides the formalism of L^p-normed modules inspired by [107], the Riemannian formulas

$$\langle \nabla_{\nabla g} X, \nabla h\rangle = \langle \nabla\langle X,\nabla g\rangle, \nabla h\rangle - \mathrm{Hess}\,(h)(X,\nabla g),$$

$$d\omega(X_1,X_2) = \langle X_1, \nabla\omega(X_2)\rangle - \langle X_2, \nabla\omega(X_1)\rangle - \omega(\nabla_X Y - \nabla_Y X)$$

which grant the possibility, as soon as one has a good definition of Hessian, to define first the covariant derivative of X and then the exterior differential of ω. The RCD assumption enters to provide good integrability estimates and non-triviality of the objects involved (for instance the existence of a rich set of $H^{2,2}(X,\mathsf{d},\mathfrak{m})$ functions). Remarkably, at the end of this process also the Hessian term in the right hand side of (5.7) is well defined, so that one can define a measure-valued Ricci tensor by $\Gamma_2(f) - \mathrm{Hess}\,(f)$ and the lower bounds on Ricci tensor can be localized.

9 Open problems

Finally, I wish to conclude this survey by stating a few open questions, on which I expect to see new developments in the near future.

• As we have seen, many equivalence and structural results of the CD theory hold under the essential non-branching assumption. At this moment, the only stable class of spaces satisfying this condition is the one of $\mathrm{RCD}(K,\infty)$ spaces. Is there a larger "non-Riemannian" stable class satisfying this condition, or how should the notion of essential non-branching be adapted to this purpose?

• Presently, as we have seen, the BE and CD theories can be closely related only in the class of infinitesimally Hilbertian m.m.s. Is there a "nonlinear" BE theory corresponding to the CD theory, without assuming Ch_2 to be quadratic? In the setting of Finsler manifolds some important steps in this direction have already been achieved, see the survey paper [84].

• In connection with Theorem 8.1, in a remarkable paper T.Colding and A.Naber [42] proved that, for Ricci limit spaces, only one of the sets $\mathcal{R}_k$

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has positive $\mathfrak{m}$-measure (so that the dimension is constant). Is this property true also for $\mathrm{RCD}^*(K,N)$ spaces?
• Even though many properties of Ricci limit spaces (i.e. limits of Riemannian manifolds) are being proved for RCD spaces, the characterization of limit spaces within RCD ones is a challenging question. Using the fact that 3-dimensional non-collapsed limits are topological manifolds [99] as well as the existence of $\mathrm{RCD}^*(0,3)$ spaces which are not topological manifolds[3], a gap between Ricci limits and RCD spaces surely exists, at least if one looks at non-collapsed limits.
• The definition of Laplacian in the metric measure setting corresponds, in the smooth setting, to the (weighted) Laplacian with homogeneous Neumann boundary conditions. For this reason the "boundary" is somehow hidden and it is not clear, not even in the RCD setting, how a reasonable definition of boundary can be given at this level of generality. A definition based on the n-dimensional Hausdorff measure of small balls, thus using only the metric structure, is proposed in [64], dealing with geodesic flow in n-dimensional Alexandrov spaces.

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