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Strong Law of large number Law of the iterated logarithm for nonlinear probabilities

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July 5, 2010





Outline

- ♦ History of LLN and LIL for probabilities
- ♦ Why to study LLN and LIL for capacities
 - Nonlinear probabilities and nonlinear expectations
- ♦ Main results

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♦ Applications



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0.1. History of LLN and LIL for probability

Law of large number(LLN):

(1) Brahmagupta (598-668), Cardano (1501-1576)

(2) Jakob Bernoulli(1713), Poisson (1835)

(3) Chebyshev, Markov, Borel(1909), Cantelli and Kolmogorov(IID).

Law of iterated logarithm(LIL):

(1) Khintchine(1924) for Bernoulli model

Kolmogorov(1929), Hartman–Wintner(1941) (IID)

(2) Levy(1937) for Martingale

(3) Strassen(1964) for functional random variables.



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(b)

0.2. Strong LLN and LIL for probabilities

Assumption: $\{X_i\}$ IID, $S_n/n := \sum_{i=1}^n X_i$, $EX_1 = \mu_i$ Then **Theorem 1:**Kolmogorov:

 $P(\lim_{n\to\infty}S_n/n=\mu)=1$

Theorem 2: Hartman–Wintner(1941): If $EX_1 = 0$, $EX_1^2 = -2^2$, Then (a)

$$P\left(\limsup_{n\to\infty}\frac{S_n}{\sqrt{2n\log\log n}}=\right) = 1$$

$$P\left(\liminf_{n\to\infty}\frac{S_n}{\sqrt{2n\log\log n}}=-\right)=1$$

(c) Suppose that $C({x_n})$ is the cluster set of a sequence of ${x_n}$ in R, then

$$P(C(\{ : S_n() / \sqrt{2n \log \log n}\}) = [-,]) = 1.$$



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0.3. Why to study LLN and LIL in Finance

THEOREM 1 (Black-Scholes, 1973:) In complete markets, there exists a unique probability measure Q, such that the pricing of option at strike date T is given by $E_Q[e^{-rT}]$. Where r = 0 is interest rate of bond.

Monte Carlo, $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = E_Q[$].

(Linear) expectation \leftarrow Black-Scholes \rightarrow Complete Markets

 $\inf_{Q \in \mathcal{P}} E_Q[], \sup_{Q \in \mathcal{P}} E_Q[] \iff$ Incomplete Markets, Q is not unique, SET \mathcal{P} .

Super-pricing: $\inf_{Q \in \mathcal{P}} E_Q[]$, $\sup_{Q \in \mathcal{P}} E_Q[]$. Nonlinear expectation! $\lim_{n \to \infty} S_n / n = ?$



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0.4. Bernoulli Trials with ambiguity

Bernoulli Trials:

Repeated independent trials are called Bernoulli trials if there are only two possible outcomes for each trial and their probabilities **REMAIN** (are no longer) the same throughout the trials.

Let $X_i = 1$ if head occurs and $X_i = 0$ if tail occurs.

$$P(X_i = 1) = , P(X_i = 0) = 1 - , S_n := \sum_{i=1}^n X_i$$

If = 1/2 (Unbalance), LLN stats

$$P\left(\lim_{n\to\infty}S_n/n=1/2\right)=1$$

Or

$$\lim_{n\to\infty}S_n/n=1/2 \quad a.s \quad (P)$$



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If a coin is balance. $P(X_i = 1) = \in [1/3, 1/2]$. Let $\mathcal{P} := \{P, \in [1/3, 1/2]\}$. $E_P[X_i] = \text{Unknown},$ But $\max_{P \in \mathcal{P}} E_P[X_i] = 1/2$, $\min_{P \in \mathcal{P}} E_P[X_i] = 1/3$. Question: what is the limit $S_n/n \rightarrow$? (a) Capacity: If $V(A) := \max_{P \in \mathcal{P}} P(A)$, $v(A) := \min_{P \in \mathcal{P}} P(A)$ Can S_n/n converge to $\max_{P \in \mathcal{P}} E_P[X_i]$ or $\min_{P \in \mathcal{P}} E_P[X_i]$ a.s. V or V? (b) The relation between the set of limit points of S_n/n and the interval of $\min_{P \in \mathcal{P}} E_P[X_i]$ and $\max_{P \in \mathcal{P}} E_P[X_i]$.



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0.5. Linear and Nonlinear Expectations

Kolmogorov: Linear expectation: $P : \mathcal{F} \to [0, 1], P(A) = E[I_A]$

 $P(A + B) = P(A) + P(B), A \cap B = \emptyset \Leftrightarrow E[+] = E[] + E[]$

Expectation is a linear functional of random variable.

Nonlinear probability(capacity): $V(\cdot) : \mathcal{F} \to [0, 1]$ but

 $V(A + B) \neq V(A) + V(B)$, even $A \cap B = \emptyset$.

Nonlinear expectation: $\mathbb{E}(\)$ is nonlinear functional in the sense of

 $\mathbb{E}[+] \neq \mathbb{E}[] + \mathbb{E}[].$

Capacity $V(A) = \mathbb{E}[I_A]$ is nonlinear.



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Modes of nonlinear expectations and capacity

(1)Choquet expectations (Choquet 1953, physics)

$$C_{V}[X] := \int_{0}^{\infty} V(X \ge t) dt + \int_{-\infty}^{0} [V(X \ge t) - 1] dt.$$

(2)g-expectation (Peng 1997)

(3) Sub-linear expectation(Peng 2007).

(a)Monotonicity: X ≥ Y implies E[X] ≥ E[Y].
(b)Constant preserving: E[c] = c, ∀c ∈ R.
(c)Sub-additivity: E[X + Y] ≤ E[X] + E[Y].
(d)Positive homogeneity: E[X] = E[X], ∀ ≥ 0.

(1) Distorted probability measure: $V(A) = g(P(A)), g : [0, 1] \rightarrow [0, 1].$ (2) 2-alternating capacity: $V(A \cup B) \leq V(A) + V(B) - V(A \cap B)$ (3) $V(A) = \max_{P \in \mathcal{P}} P(A), \mathcal{P}$ set of Probability.

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3. Definition: capacity and nonlinear expectation

(1) Probability space : $(, \mathcal{F}, P) \Rightarrow (, \mathcal{F}, P)$. Where $\mathcal{P} := \{P : \in \}$. (2) Capacity: $P \Rightarrow (v, V)$, where

$$V(A) = \inf_{Q \in \mathcal{P}} Q(A), \quad V(A) = \sup_{Q \in \mathcal{P}} Q(A).$$

(3)Property:

 $V(A) + V(A^{c}) \geq 1$, $V(A) + V(A^{c}) \leq 1$

but

$$V(A) + V(A^{c}) = 1.$$

(4) Nonlinear expectations: Lower-upper expectation $\mathcal{E}[]$ and $\mathbb{E}[]$

$$\mathcal{E}[] = \inf_{Q \in \mathcal{P}} E_Q[], \qquad \mathbb{E}[] = \sup_{Q \in \mathcal{P}} E_Q[]$$

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V(AB) = V(A)V(B), v(AB) = v(A)v(B)

Theorem (Epstein, 02, Marinacci, 99, 05). Bounded, Polish, $C_{V}[X_{i}] = \mu$, $C_{V}[X_{i}] = \overline{\mu}$. { X_{i} } IID, then

$$v(\underline{\mu} \leq \liminf_{n \to \infty} S_n / n \leq \limsup_{n \to \infty} S_n / n \leq \overline{\mu}\}) = 1.$$

Where *V* is totally 2-alternating $V(A \bigcup B) \le V(A) + V(B) - V(AB)$, here C_V and C_V is Choquet are integrals. Note $C_V[X] < \mathcal{E}[X] < \mathbb{E}[X] < C_V[X], \forall X$.



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4.1. Limit theorem 1

Theorem: If $\{X_i\}$ is IID, then $\frac{S_n}{n}$ converges as $n \to \infty$ a.s. V if and only if $\mathcal{E}[X_1] = \mathbb{E}[X_1].$

In this case,

 $\lim S_n/n = \mathcal{E}[X_1], \quad a.s. \quad v.$





THEOREM 3 $\{X_i\}_{i=1}^n$ IID under nonlinear expectation \mathbb{E} . Set $\mu := \mathbb{E}[X_i]$, $\mu := \mathcal{E}[X_i]$ and $S_n := \sum_{i=1}^n X_i$. If $\mathbb{E}[|X_i|^{1+}] < \infty$ for > 0. Then

 $v (\in : \underline{\mu} \le \liminf_{n \to \infty} S_n() / n \le \limsup_{n \to \infty} S_n / n() \le \overline{\mu}) = 1.$

(11)

(1)

5. Main results

$$V (\in : \limsup_{n \to \infty} S_n()/n = \overline{\mu}) = 1$$
$$V (\in : \liminf_{n \to \infty} S_n()/n = \underline{\mu}) = 1.$$

(111) Suppose that $C(\{S_n()/n\})$ is the cluster set of a sequence of $\{S_n()/n\}$, then

 $V(\in C(\{S_n(n)/n\}) = [\underline{\mu}, \overline{\mu}]) = 1$



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(1)

(II)

6. Law of iterated logarithm for sub-linear expectations

THEOREM 4 { X_n } bounded IID. $\mathbb{E}[X_1] = \mathcal{E}[X_1] = 0, -2 := \mathbb{E}[X_1^2], -2 := \mathcal{E}[X_1^2]$. Let $S_n := \sum_{i=1}^n X_i, a_n := \sqrt{2n \lg \lg n}$, then

$$Y\left(-\leq \limsup_{n} \frac{S_n}{a_n} \leq -\right) = 1;$$

$$\sqrt{\left(-- \leq \liminf_{n} \frac{S_n}{a_n} \leq -\right)} = 1.$$

(III) Suppose that $C(\{x_n\})$ is the cluster set of a sequence of $\{x_n\}$ in R, then

$$/\left(C(\{S_n/\sqrt{2n\log\log n}\}) \supset (-_,_)\right) = 1.$$

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7. Key of proof

THEOREM 5 Suppose is distributed to G normal $N(0; [_^2, -^2])$, where $0 < _ \le - < \infty$. Let be a bounded continuous function. Furthermore, if is a positively even function, then, for any $b \in R$,

 $e^{-\frac{b^2}{2-2}}\mathcal{E}[(-)] \leq \mathcal{E}[(-b)].$



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8. Application

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Total 100 balls in box, Black + Red + Yellow = 100, Black = Red, Yellow \in [30, 40], then $P_Y \in$ [3/10, 4/10]. Take a ball from this box, $X_i = 1$, if ball is black, $X_i = 0$, if ball is Yellow, $X_i = -1$ for red. $S_n = \sum_{i=1}^n X_i$, is the excess frequency of black than Red Then (a) $\mathbb{E}[X_i] = \mathcal{E}[X_i] = 0$

$$\sqrt{6/10} \le \lim \sup_{n \to \infty} \frac{S_n}{\sqrt{2n \lg \lg n}} \le \sqrt{7/10}.$$









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Thank you !