Stochastic flows generated by SDE.

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This talk is based on a joint work with Dejun Luo and Anton Thalmaier.

Let's give a brief review on ODE and make comparison with SDE. We consider the ODE on $\mathbb{R}^d$

$$\frac{dX_t}{dt} = V(X_t), X_0 = x, \tag{1}$$

If the coefficients A_i are globally Lipschitz continuous, then we can also solve (2) by Picard iteration. By moment estimate and Kolmogoroff modification theorem, it was proved by H. Kunita that the SDE (2) defines a stochastic flow of homeomorphisms of $\mathbb{R}^d$: if $(\Omega, \mathbf{P})$ denotes the probability space on which the Brownian motion is defined, then there exists a full measure subset $\Omega_0 \subset \Omega$ such that for $w \in \Omega_0$, for each $t > 0$, $x \to X_t(w, x)$ is a global homeomorphism of $\mathbb{R}^d$. However in contrast with ODE, the regularity of the homeomorphism X_t is only Hölder continuity of order $0 < \alpha < 1$. Thus it is not clear whether the Lebesgue measure on $\mathbb{R}^d$ admits a density under the flow X_t.

If V is smooth such that the lift time $\tau_x = +\infty$ for each $x \in \mathbb{R}^d$, then $x \to X_t(x)$ is a global diffeomorphism of $\mathbb{R}^d$. But for SDE, we have the notion of *completeness* and *strong completeness*.
Now let $\theta \in C^1(\mathbb{R}^d)$, then the function $u_t(x) = \theta(X_t^{-1}(x))$ satisfies the PDE

$$\frac{du_t}{dt} + V \cdot \nabla u_t = 0. \tag{3}$$

- When V satisfies Osgood conditions

$$|V(x) - V(y)| \le C\,|x - y| \log \frac{1}{|x - y|},\ |x - y| \le \delta_0, \tag{4}$$

the ODE (1) still defines a flow of homeomorphisms and $u_t(x) = \theta(X_t^{-1}(x))$ for $\theta \in C(\mathbb{R}^d)$ solves (3) in distribution sense, not necessarily uniquely. But it allows to prove that if $\mathrm{div}(V) \in L^\infty$ exists, then the Lebesgue measure is quasi-invariant under the flow X_t.

• When $V \in W^p_{1,loc}$, then the transport equation

$$\frac{du_t}{dt} + V \cdot \nabla u_t = 0.$$

admits a unique solution $u \in L^\infty([0,T], L^p(\mathbb{R}^d))$ if $\mathrm{div}(V) \in L^\infty$ and V has linear growth. Moreover, the following property holds: *if u is a solution to* (3)*, then for any $\beta \in C^1_b(\mathbb{R})$, $\beta(u)$ is still a solution to* (3)*, but with a di®erent initial data.*

This is a key point which allowed Di Perna and Lions to solve

$$X_t(x) = x + \int_0^t V(X_s(x))\,ds;$$

there exists a unique flow of measurable maps $X_t : \mathbb{R}^d \to \mathbb{R}^d$ such that $(X_t)_*(dx) = K_t\,dx$ and the above equality holds a.e.

Stochastic transport equations have been considered by B. L. Rozovskii. When the drift term A_0 is bounded and satisfies the Osgood condition (4), and the diffusion coefficents $A_1, \cdots, A_m \in C_b^4$, the Stratanovich SDE

$$dX_t = \sum_{i=1}^{m} A_i(X_t) \circ dw_t^i + A_0(X_t)\, dt$$

defines a flow of homeomorphisms X_t, which leave the Lebesgue quasi-invariant if $\mathrm{div}(A_0) \in L^\infty$ exists (Luo, Bull. Sci. Math. 2009).
Instead of Osgood condition, if $A_0 \in W^p_{1,loc}$, the above Di Perna-Lions method does not work well, due to stochastic contraction: the invariance under β fails to hold.

We consider the standard Gaussian measure as initial measure $\gamma_d(dx)$:

$$\gamma_d(dx) = \frac{e^{-|x|^2/2}}{(\sqrt{2\pi})^d}\,dx.$$

Then in the case where V is smooth, the push forward measure $(X_t)_\sharp\gamma_d$ admits the density K_t with respect to γ_d and

$$K_t(x) = \exp\left(\int_0^t -\mathrm{div}_\gamma(V)(X_{-s}(x))ds\right),$$

and the Cruzeiro's estimate in $L^p(\gamma_d)$ for $p > 1$

$$||K_t||_{L^p}^p \leq \int_{\mathbb{R}^d} \exp\left(\frac{p^2 t}{p-1}|\mathrm{div}_\gamma(V)|\right) d\gamma_d$$

holds, where $\mathrm{div}_\gamma(V) = \sum_{i=1}^d (\partial V^i/\partial x_i - x_i V^i)$.

For SDE

$$dX_t = \sum_{j=1}^{m} A_j(X_t)\, dw_t^j + A_0(X_t)\, dt, \quad X_0 = x, \tag{5}$$

if $A_j, j = 0, 1, \ldots, m$, are in $C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$, the SDE (5) defines a flow of diffeomorphisms, and Kunita showed that the measures on $\mathbb{R}^d$ which have a strictly positive smooth density with respect to Lebesgue measure are quasi-invariant under the flow. If we consider the standard Gaussian measure γ_d, then the density $\tilde{K}_t(w, x)$ of $(X_t^{-1}(w, \cdot))_\sharp \gamma_d$ with respect to γ_d admits explicit expression

$$\tilde{K}_t(x) = \exp\left(\sum_{j=1}^{m} \int_0^t \operatorname{div}_\gamma(A_j)(X_s(x)) \circ dw_s^j + \int_0^t \operatorname{div}_\gamma(\tilde{A}_0)(X_s(x))\, ds \right),$$

where $\circ dw_s^j$ denotes the Stratanovich stochastic integral and

$$\tilde{A} = A_0 - \frac{1}{2}\sum_{j=1}^{d} \mathcal{L}_{A_j} A_j.$$

Theorem (A)

Let $K_t(w, x)$ be the density of $(X_t)_\sharp \gamma_d$ with respect to γ_d. Then for $p > 1$, we have

$$\|K_t\|_{L^p(\mathbf{P}\times\gamma_d)} \le \left(\int_{\mathbb{R}^d} \exp\left(pt \left[2|\mathrm{div}_\gamma(A_0)| + \sum_{j=1}^{m} \left(|A_j|^2 + |\nabla A_j|^2 + 2(p-1)|\mathrm{div}_\gamma(A_j)|^2 \right) \right] \right) d\gamma_d \right)^{\frac{p-1}{p(2p-1)}}.$$

We have no explicit expression for K_t, but its L^p estimate is easier than $\tilde{K}_t$. In fact, we have the relation

$$K_t(x) = 1/\tilde{K}_t\left(X_t^{-1}(x)\right).$$

$$
\begin{aligned}
\int_{\mathbb{R}^d} \mathbb{E}[K_t^p(x)]\, d\gamma_d(x) &= \mathbb{E} \int_{\mathbb{R}^d} \left[\tilde{K}_t \left(X_t^{-1}(x) \right) \right]^{-p} d\gamma_d(x) \\
&= \mathbb{E} \int_{\mathbb{R}^d} \left[\tilde{K}_t(y) \right]^{-p} \tilde{K}_t(y)\, d\gamma_d(y) \\
&= \int_{\mathbb{R}^d} \mathbb{E} \left[\left(\tilde{K}_t(x) \right)^{-p+1} \right] d\gamma_d(x).
\end{aligned}
$$

Transfering Stratanovich integrals to Ito's one, we have

$$
\begin{aligned}
\tilde{K}_t(x) = \exp \Bigg(& - \sum_{j=1}^{m} \int_0^t \mathrm{div}_\gamma(A_j)(X_s(x))\, dw_s^j \\
& - \int_0^t \left(\frac{1}{2} \sum_{j=1}^{m} \mathcal{L}_{A_j} \mathrm{div}_\gamma(A_j) + \mathrm{div}_\gamma(\tilde{A}_0) \right) (X_s(x))\, ds \Bigg).
\end{aligned}
$$

But the following commutation formula holds

$$\frac{1}{2}\sum_{j=1}^{m}\mathcal{L}_{A_j}\mathrm{div}_\gamma(A_j)+\mathrm{div}_\gamma(\tilde{A}_0)$$
$$=\mathrm{div}_\gamma(A_0)+\frac{1}{2}\sum_{j=1}^{m}|A_j|^2+\frac{1}{2}\sum_{j=1}^{m}\langle\nabla A_j,(\nabla A_j)^*\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of $\mathbb{R}^d\otimes\mathbb{R}^d$ and $(\nabla A_j)^*$ the transpose of ∇A_j. Now using exponential martingales and Cruzeiro's method, we get the result.

Theorem (B)

Let $A_0, A_1, \ldots, A_m$ be continuous vector fields on $\mathbb{R}^d$ of linear growth. Assume that the diffusion coefficients $A_1, \ldots, A_m$ are in the Sobolev space $\bigcap_{q>1} \mathbb{D}_1^q(\gamma_d)$ and that $\mathrm{div}_\gamma(A_0)$ exists; furthermore there exists a constant $\lambda_0 > 0$ such that

$$\int_{\mathbb{R}^d} \exp\left[\lambda_0 \left(|\mathrm{div}_\gamma(A_0)| + \sum_{j=1}^m \big(|\mathrm{div}_\gamma(A_j)|^2 + |\nabla A_j|^2 \big) \right)\right] d\gamma_d < +\infty. \tag{6}$$

Suppose that pathwise uniqueness holds for

$$dX_t = \sum_{j=1}^m A_j(X_t)\, dw_t^j + A_0(X_t)\, dt, \quad X_0 = x.$$

Then $(X_t)_\# \gamma_d$ is absolutely continuous with respect to γ_d and the density is in $L^1 \log L^1$.

Beyond the Lipschitz condition, several sufficient conditions guaranteeing pathwise uniqueness for SDE can be found in the literature. For example in: S. Fang, T. Zhang, *A study of a class of stochastic differential equations with non-Lipschitzian coefficients.* Probab. Theory Related Fields 132 (2005), 356–390, the authors give the condition

$$\sum_{j=1}^{m} |A_j(x) - A_j(y)|^2 \leq C\,|x-y|^2 r\big(|x-y|^2\big),$$
$$|A_0(x) - A_0(y)| \leq C\,|x-y| r\big(|x-y|^2\big),$$

for $|x-y| \leq c_0$ small enough, where $r\colon\ ]0,c_0] \to]0,+\infty[$ is C^1 satisfying

$$\int_0^{c_0} \frac{ds}{sr(s)} = +\infty.$$

In order to apply the a priori estimate for density, we first have to restrict on a small interval $[0, T_0]$; beyond T_0, we utilize the property of flow: using again the relation

$$K_t(x) = 1/\tilde{K}_t(X_t^{-1}(x)), \quad \text{or} \quad K_t(X_t(x)) = 1/\tilde{K}_t(x),$$

it is possible to estimate

$$\mathbb{E}\int_{\mathbb{R}^d} |\log K_t|\, K_t\, d\gamma_d = \mathbb{E}\int_{\mathbb{R}^d} |\log \tilde{K}_t(x)|\, d\gamma_d(x)$$

by

$$\Lambda_{T_0} := \int_{\mathbb{R}^d} \exp\left(4T_0\left[|A_0| + e|\mathrm{div}_\gamma(A_0)| + \sum_{j=1}^m \left[4|A_j|^2 + |\nabla A_j|^2 + 2e^2|\mathrm{div}_\gamma(A_j)|^2\right]\right]\right) d\gamma_d < \infty.$$

A consequence of above theorem concerns the following classical situation.

Theorem (C)

Let $A_0, A_1, \ldots, A_m$ be globally Lipschitz continuous. Suppose that there exists a constant $C > 0$ such

We give the following example where the density is strictly positive.

Theorem

Let $A_1, \ldots, A_m$ be bounded C^2 vector ¯elds such that their derivatives up to order 2 *grow at most linearly, and let A_0 be a continuous vector ¯eld of linear growth. Suppose that*

$$|A_0(x) - A_0(y)| \leq C_R\, |x - y| \log_k \frac{1}{|x - y|}$$

for $|x| \leq R,\ |y| \leq R,\ |x - y| \leq c_0$ where $\log_k s = (\log s)(\log \log s) \ldots (\log \ldots \log s)$. *Suppose further that* $\mathrm{div}(A_0)$ *exists and is bounded. Then the stochastic °ow X_t de¯ned by* SDE (5) *leaves the Lebesgue measure quasi-invariant.*

Note that for SDE , even for vector fields $A_0, A_1, \ldots, A_m$ in C^∞ with linear growth, if no conditions were imposed on the growth of the derivatives, the SDE (5) may not define a flow of diffeomorphisms. More precisely, let τ_x be the life time of the solution starting from x. The SDE (5) is said to be *complete* if for each $x \in \mathbb{R}^d$, $\mathbf{P}(\tau_x = +\infty) = 1$; it is said to be *strongly complete* if $\mathbf{P}(\tau_x = +\infty, x \in \mathbb{R}^d) = 1$. There are examples for which the coefficients are smooth, but such that the SDE (5) is not strongly complete. Under the growth of order $\log R$ on derivatives, it was proved that $x \to X_t(w, x)$ is a global of homeomorphism. Under the hypothesis of above theorem, we do not know if the SDE defines a flow of homeomorphisms.

However there exists a full measure subset $\Omega_0 \subset \Omega$ such that for all $w \in \Omega_0$, $\tau_x(w) = +\infty$ holds for μ-almost every $x \in \mathbb{R}^d$. Now under the existence of a complete unique strong solution to SDE (5), we have a flow of measurable maps $x \to X_t(w, x)$.

Now consider the case where $A_0 \in \mathbb{D}_1^q(\mathbb{R}^d)$ for some $q > 1$, without being continuous.

We say that a measurable map $X : \Omega \times \mathbb{R}^d \to C([0,T], \mathbb{R}^d)$ is a solution to the Itô SDE

$$dX_t = \sum_{i=1}^{m} A_i(X_t)\, dw_t^i + A_0(X_t)\, dt, \quad X_0 = x,$$

if

(i) for each $t \in [0,T]$ and almost all $x \in \mathbb{R}^d$, $w \to X_t(w,x)$ is measurable with respect to $\mathcal{F}_t$, i.e., the natural filtration generated by the Brownian motion $\{w_s : s \le t\}$;

(ii) for each $t \in [0,T]$, there exists $K_t \in L^1(\mathbf{P} \times \mathbb{R}^d)$ such that $(X_t(w,\cdot))_{\#}\gamma_d$ admits K_t as the density with respect to γ_d;

(iii) almost surely

$$\sum_{i=1}^{m} \int_0^T |A_i(X_s(w,x))|^2\, ds + \int_0^T |A_0(X_s(w,x))|\, ds < +\infty;$$

(iv) for almost all $x \in \mathbb{R}^d$,

$$X_t(w,x) = x + \sum_{i=1}^{m} \int_0^t A_i(X_s(w,x))\, dw_s^i + \int_0^t A_0(X_s(w,x))\, ds;$$

(v) the flow property holds

$$X_{t+s}(w,x) = X_t(\theta_s w, X_s(w,x)).$$

Theorem (C)

Assume that the diffusion coefficients $A_1, \ldots, A_m$ *belong to the Sobolev space* $\bigcap_{q>1} \mathbb{D}_1^q(\gamma_d)$ *and the drift* $A_0 \in \mathbb{D}_1^q(\gamma_d)$ *for some* $q > 1$*. Assume*

$$\int_{\mathbb{R}^d} \exp\left[\lambda_0 \left(|\mathrm{div}_\gamma(A_0)| + \sum_{j=1}^{m} \big(|\mathrm{div}_\gamma(A_j)|^2 + |\nabla A_j|^2 \big) \right)\right] d\gamma_d < +\infty,$$

and that the coefficients $A_0, A_1, \ldots, A_m$ *are of linear growth, then there is a unique stochastic flow of measurable maps* $X : [0, T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$*, which solves* (5) *for almost all initial* $x \in \mathbb{R}^d$ *and the push-forward* $(X_t(w, \cdot))_{\#}\gamma_d$ *admits a density with respect to* γ_d*, which is in* $L^1 \log L^1$*.*

Lemma (D)

Let $q > 1$. Suppose that $A_1, \ldots, A_m$ as well as $\hat{A}_1, \ldots, \hat{A}_m$ are in $\mathbb{D}_1^{2q}(\gamma_d)$ and $A_0, \hat{A}_0 \in \mathbb{D}_1^q(\gamma_d)$. Then, for any $T > 0$ and $R > 0$, there exist constants $C_{d,q,R} > 0$ and $C_T > 0$ such that for any $\sigma > 0$,

$$
\begin{aligned}
&\mathbb{E}\left[\int_{G_R} \log\left(\frac{\sup_{0\le t\le T}|X_t - \hat{X}_t|^2}{\sigma^2} + 1\right) d\gamma_d\right] \\
&\le C_T \Lambda_{p,T} \left\{ C_{d,q,R}\left[\|\nabla A_0\|_{L^q} + + \sum_{i=1}^m \|\nabla A_i\|_{L^{2q}}^2\right] \right. \\
&\quad \left. + \frac{1}{\sigma^2}\sum_{i=1}^m \|A_i - \hat{A}_i\|_{L^{2q}}^2 + \frac{1}{\sigma}\left[\|A_0 - \hat{A}_0\|_{L^q}\right] \right\},
\end{aligned}
$$

where p is the conjugate number of q: $1/p + 1/q = 1$, and

$$
G_R(w) = \left\{ x \in \mathbb{R}^d : \sup_{0\le t\le T} |X_t(w,x)| \vee |\hat{X}_t(w,x)| \le R \right\}. \qquad (8)
$$

Let X^n be the solution associated to A_j^ε with $\varepsilon = 1/n$. Let

$$\sigma_{n,k} = ||A_0^n - A_0^k||_{L^q} + \left(\sum_{i=1}^{m} ||A_i^n - A_i^k||^2_{L^{2q}} \right)^{1/2}.$$

By above result,

$$I_{n,k} := \mathbb{E}\left[\int_{G_{n,R} \cap G_{k,R}} \log\left(\frac{\|X^n - X^k\|^2_{\infty, T_0}}{\sigma^2_{n,k}} + 1 \right) d\gamma_d \right]$$

is bounded with respect to n, k, where $|| \cdot ||_{\infty, T_0}$ denotes the uniform norm over $[0, T_0]$.

Proof of lemma D:

let $\xi_t = X_t - \hat{X}_t$, then $\xi_0 = 0$. By Itô's formula,

$$d|\xi_t|^2 = 2\sum_{i=1}^{m} \langle \xi_t, A_i(X_t) - \hat{A}_i(\hat{X}_t)\rangle\, dw_t^i + 2\langle \xi_t, A_0(X_t) - \hat{A}_0(\hat{X}_t)\rangle\, dt$$
$$+ \sum_{i=1}^{m} |A_i(X_t) - \hat{A}_i(\hat{X}_t)|^2\, dt,$$

and

$$\begin{aligned} & d\log(|\xi_t|^2 + \sigma^2) \\ = {} & 2\sum_{i=1}^{m} \frac{\langle \xi_t, A_i(X_t) - \hat{A}_i(\hat{X}_t)\rangle}{|\xi_t|^2 + \sigma^2}\, dw_t^i + 2\frac{\langle \xi_t, A_0(X_t) - \hat{A}_0(\hat{X}_t)\rangle}{|\xi_t|^2 + \sigma^2}\, dt \\ & + \sum_{i=1}^{m} \frac{|A_i(X_t) - \hat{A}_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2}\, dt - 2\sum_{i=1}^{m} \frac{\langle \xi_t, A_i(X_t) - \hat{A}_i(\hat{X}_t)\rangle^2}{(|\xi_t|^2 + \sigma^2)^2}\, dt. \end{aligned}$$

Let $\tau_R(x) = \inf\{t \ge 0\colon |X_t(x)| \vee |\hat{X}_t(x)| > R\}$. Remark that almost surely, $G_R \subset \{x\colon \tau_R(x) > T\}$ and for any $t \ge 0$, $\{\tau_R > t\} \subset B(R)$. We can estimate the martingale term by

$$\int_0^T \left(\mathbb{E} \int_{\{\tau_R > t\}} \sum_{i=1}^m \frac{|A_i(X_t) - \hat{A}_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} d\gamma_d \right) dt.$$

We have $A_i(X_t) - \hat{A}_i(\hat{X}_t) = A_i(X_t) - A_i(\hat{X}_t) + A_i(\hat{X}_t) - \hat{A}_i(\hat{X}_t)$. Using the density $\hat{K}_t$, it is clear that

$$\begin{aligned}
&\mathbb{E} \int_{\{\tau_R > t\}} \frac{|A_i(\hat{X}_t) - \hat{A}_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2}\, d\gamma_d \\
&\le \frac{1}{\sigma^2}\, \mathbb{E} \int_{\mathbb{R}^d} |A_i(\hat{X}_t) - \hat{A}_i(\hat{X}_t)|^2\, d\gamma_d \\
&= \frac{1}{\sigma^2}\, \mathbb{E} \int_{\mathbb{R}^d} |A_i - \hat{A}_i|^2 \hat{K}_t\, d\gamma_d.
\end{aligned}$$

Note that on the set $\{\tau_R > t\}$, $X_t, \hat{X}_t \in B(R)$, thus $|X_t - \hat{X}_t| \le 2R$. Since $(X_t)_{\#}\gamma_d \ll \gamma_d$ and $(\hat{X}_t)_{\#}\gamma_d \ll \gamma_d$, we can apply

$$|f(x) - f(y)| \le C_d \, |x - y| \, \big(M_R|\nabla f|(x) + M_R|\nabla f|(y) \big)$$

so that

$$|A_i(X_t) - A_i(\hat{X}_t)| \le C_d |X_t - \hat{X}_t| \, \big(M_{2R}|\nabla A_i|(X_t) + M_{2R}|\nabla A_i|(\hat{X}_t) \big).$$

Therefore

$$\begin{aligned}
&\mathbb{E} \left[\int_{\{\tau_R > t\}} \frac{|A_i(X_t) - A_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} \, d\gamma_d \right] \\
&\le 2C_d^2 \, \mathbb{E} \int_{B(R)} (M_{2R}|\nabla A_i|)^2 \big(K_t + \hat{K}_t \big) \, d\gamma_d \\
&\le 4C_d^2 \Lambda_{p,T} \left(\int_{B(R)} \big(M_{2R}|\nabla A_i| \big)^{2q} \, d\gamma_d \right)^{1/q}.
\end{aligned}$$

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Finally we would like to mention some related works under weaker Sobolev type conditions:

• the connection between weak solutions and Fokker-Planck equations has been investigated by A. Figalli, Lebris and Lions;
• some notions of "generalized solutions", as well as the phenomena of coalescence and splitting, have been explored by LeJan and Raimond1;
• stochastic transport equations are studied by Flandoli, Gubinelli and Priola.

Here ara some more references:

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