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Extensions of a theorem of Hsu and Robbins on the convergence rates in the law of large numbers

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1 Introduction

1.1 Convergence rate in the law of large numbers: the iid case

Consider i.i.d. r.v. X_j with $EX_j = 0$. Let

$$S_n = X_1 + \dots + X_n.$$

Law of Large numbers:

$$\frac{S_n}{n} \rightarrow 0:$$

Question: at what rate $P(|S_n| > n'') \rightarrow 0$?

The theorem of Hsu-Robbins-Erdos

Hsu and Robbins (1947):

$$EX_1^2 < \infty \Rightarrow \sum_n P(|S_n| > n'') < \infty \quad \forall'' > 0:$$

("complete convergence", which implies a.s. convergence)

Erdos (1949): the converse also holds:

$$EX_1^2 < \infty \Leftarrow \sum_n P(|S_n| > n'') < \infty \quad \forall'' > 0:$$

Spitzer (1956):

$$\sum_n n^{-1} P(|S_n| > n'') < \infty \quad \forall'' > 0 \text{ whenever } EX_1 = 0:$$

Baum and Katz (1965): for $p > 1$;

$$E|X_1|^p < \infty \Leftrightarrow \sum_n n^{p-2} P(|S_n| > n'') < \infty \quad \forall'' > 0;$$

in particular,

$$E|X_1|^p < \infty \Rightarrow P(|S_n| > n'') = o(n^{-(p-1)})$$

Question: is it valid for martingale differences?

1.2 Convergence rates in the law of large numbers: the martingale case

Is the theorem of Baum and Katz (1965) still valid for martingale differences (X_j)?

$$\{\emptyset; \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$$

$\forall j$, X_j are \mathcal{F}_j measurable with $E[X_j | \mathcal{F}_{j-1}] = 0$

($\Leftrightarrow S_n = X_1 + \dots + X_n$ is a martingale.)

Lesigne and Volney (2001): $p \geq 2$

$$E|X_1|^p < \infty \Rightarrow P(|S_n| > n^\alpha) = o(n^{-p/2})$$

and the exponent $p/2$ is the best possible, even for stationary and ergodic sequences of martingale differences.

Therefore the theorem of Baum and Katz does not hold for martingale differences without additional conditions.

[Curiously, Stoica (2007) claimed that the theorem of Baum and Katz still holds for $p > 2$ in the case of martingale differences without additional assumption. His claim is a contradiction with the conclusion of Lesigne and Volney (2001), and his proof is wrong: **he chose an element in an empty set!**]

1.3 Under what conditions the theorem of Baum and Katz still holds for martingale differences?

Alsmeyer (1990) proved that the theorem of Baum and Katz of order $p > 1$ still holds for martingale differences (X_j) if for some $\alpha \in (1; 2]$ and $q > (p - 1)/(\alpha - 1)$,

$$\sup_{n \geq 1} \left\| \frac{1}{n} \sum_{j=1}^n E[X_j | \mathcal{F}_{j-1}] \right\|_q < \infty$$

where $\|\cdot\|_q$ denotes the L^q norm.

His result is already nice, but:

Our objective: extend the theorem of Baum and Katz (1965) to a large class of martingale arrays, in improving Alsmeyer's result for martingales, by establishing a sharp comparison result between

$$P\left(\sum_{j=1}^{\infty} X_{n;j} > \epsilon\right) \text{ and } \sum_{j=1}^{\infty} P(X_{n;j} > \epsilon)$$

for arrays of martingale differences $\{X_{n;j} : j \geq 1\}$.

Our result is sharper than the known ones even in the independent (not necessarily identically distributed) case.

2. Main results for martingale arrays

For $n \geq 1$, let $\{(X_{nj}; \mathcal{F}_{nj}) : j \geq 1\}$ be a sequence of martingale differences, and write

$$m_n(\cdot) = \sum_{j=1}^{\infty} \mathbb{E}[|X_{nj}| | \mathcal{F}_{n;j-1}]; \quad \cdot \in (1; 2];$$

$$S_{n;j} = \sum_{i=1}^j X_{ni}; \quad j \geq 1;$$

$$S_{n;\infty} = \sum_{i=1}^{\infty} X_{ni};$$

Lemma 1 (Law of large numbers) If for some $p \in (1; 2]$,

$$E m_n(\epsilon) := \sum_{j=1}^{\infty} E[|X_{nj}|^p] \rightarrow 0;$$

then for all $\epsilon > 0$,

$$P\left\{\sup_{j \geq 1} |S_{n;j}| > \epsilon\right\} \rightarrow 0$$

and

$$P\{|S_{n;\infty}| > \epsilon\} \rightarrow 0:$$

We are interested in their convergence rates.

Theorem 1 Let $\Phi : \mathbb{N} \mapsto [0; \infty)$. Suppose that for some $\epsilon \in (1; 2]$; $q \in [1; \infty)$ and $\epsilon_0 \in (0; 1)$,

$$\mathbb{E} m_n^q(\epsilon) \rightarrow 0 \text{ and } \sum_{n=1}^{\infty} \Phi(n) (\mathbb{E} m_n^q(\epsilon))^{1-\epsilon_0} < \infty: \quad (C1)$$

Then the following assertions are all equivalent:

$$\sum_{n=1}^{\infty} \Phi(n) \sum_{j=1}^{\infty} P\{|X_{nj}| > \epsilon\} < \infty \quad \forall \epsilon > 0; \quad (1)$$

$$\sum_{n=1}^{\infty} \Phi(n) P\{\sup_{j \geq 1} |S_{nj}| > \epsilon\} < \infty \quad \forall \epsilon > 0; \quad (2)$$

$$\sum_{n=1}^{\infty} \Phi(n) P\{|S_{n;\infty}| > \epsilon\} < \infty \quad \forall \epsilon > 0: \quad (3)$$

Remark. The condition (C1) holds if for some $r \in \mathbb{R}$ and $\alpha_1 > 0$,

$$\Phi(n) = O(n^r) \text{ and } \|m_n(\cdot)\|_\infty = O(n^{-\alpha_1}): \quad (C1')$$

In the case where this holds with $\alpha_1 = 2$, Ghosal and Chandra (1998) proved that (1) implies (2); our result is sharper because we have the equivalence.

Theorem 2 Let $\Phi : \mathbb{N} \mapsto [0; \infty)$ be such that $\Phi(n) \rightarrow \infty$. Suppose that for some $\alpha \in (1; 2]$; $q \in [1; \infty)$ and $\alpha_0 \in (0; 1)$,

$$\Phi(n)(\mathbb{E} m_n^q(\cdot))^{1-\alpha_0} = o(1) \quad (\text{resp: } O(1)) : \quad (C2)$$

3. Consequences for martingales We now consider the single martingale case

$$S_j = X_1 + \dots + X_j$$

w.r.t. a filtration

$$\{\emptyset; \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$$

By definition, $E[X_j | \mathcal{F}_{j-1}] = 0$.

For simplicity, let us only consider the case where

$$\Phi(n) = n^{p-2} \cdot (n);$$

where $p > 1$, \cdot is a function slowly varying at ∞ :

$$\lim_{x \rightarrow \infty} \frac{\cdot(x)}{\cdot(x)} = 1 \quad \forall \quad > 0:$$

Notice that

$$S_n/n \rightarrow 0 \text{ a.s. iff } P\left(\sup_{j \geq n} \frac{|S_j|}{j} > \epsilon\right) \rightarrow 0 \forall \epsilon > 0:$$

To consider its rate of convergence, we shall use the condition that for some $p \in (1; 2]$ and $q \in [1; \infty)$ with $q > (p - 1)/(p - 1)$,

$$\sup_{n \geq 1} \|\underline{m}_n(\cdot; n)\|_q < \infty; \quad (C3)$$

where $\underline{m}_n(\cdot; n) = \frac{1}{n} \sum_{j=1}^n E[|X_j| | \mathcal{F}_{j-1}]$. Remark that (C3) holds evidently if for some constant $C > 0$ and all $j \geq 1$,

$$E[|X_j| | \mathcal{F}_{j-1}] \leq C \text{ a.s.} \quad (C4)$$

Theorem 3 Let $p > 1$ and $\psi \geq 0$ be slowly varying at ∞ . Under (C3) or (C4), the following assertions are equivalent:

$$\sum_{n=1}^{\infty} n^{p-2} \psi(n) \sum_{j=1}^n P\{|X_j| > n''\} < \infty \quad \forall'' > 0; \quad (7)$$

$$\sum_{n=1}^{\infty} n^{p-2} \psi(n) P\left\{\sup_{1 \leq j \leq n} |S_j| > n''\right\} < \infty \quad \forall'' > 0; \quad (8)$$

$$\sum_{n=1}^{\infty} n^{p-2} \psi(n) P\{|S_n| > n''\} < \infty \quad \forall'' > 0; \quad (9)$$

$$\sum_{n=1}^{\infty} n^{p-2} \psi(n) P\left\{\sup_{j \geq n} \frac{|S_j|}{j} > ''\right\} < \infty \quad \forall'' > 0; \quad (10)$$

Remark. If X_j are identically distributed, then (7) is equivalent to the moment condition

$$E|X_1|^{p_\gamma(|X_1|)} < \infty:$$

So Theorem 3 is an extension of the result of Baum and Katz (1965). When γ is a constant, it was proved by Alsmeyer (1991).

Theorem 4 Let $p > 1$ and $\gamma \geq 0$ be slowly varying at ∞ . Under (C3) or (C4), the following assertions are equivalent:

$$n^{p-1-\gamma}(n) \sum_{j=1}^n P\{|X_j| > n''\} = o(1) \quad (\text{resp: } O(1)) \quad \forall'' > 0; \quad (11)$$

$$n^{p-1-\gamma}(n) P\left\{ \sup_{1 \leq j \leq n} |S_j| > n'' \right\} = o(1) \quad (\text{resp: } O(1)) \quad \forall'' > 0; \quad (12)$$

$$n^{p-1-\gamma}(n) P\{|S_n| > n''\} = o(1) \quad (\text{resp: } O(1)) \quad \forall'' > 0: \quad (13)$$

$$n^{p-1-\gamma}(n) P\left\{ \sup_{j \geq n} \frac{|S_j|}{j} > '' \right\} = o(1) \quad (\text{resp: } O(1)) \quad \forall'' > 0: \quad (14)$$

4. Applications to sums of weighted random variables.

Example: Cesàro summation for martingale differences.

For $a > -1$, let $A_0^a = 1$ and

$$A_n^a = \frac{(a+1)(a+2)\cdots(a+n)}{n!}; \quad n \geq 1:$$

Then $A_n^a \sim \frac{n^a}{\Gamma(a+1)}$ as $n \rightarrow \infty$; and $\frac{1}{A_n^a} \sum_{j=0}^n A_{n-j}^{a-1} = 1$. We consider convergence rates of

$$\frac{\sum_{j=0}^n A_{n-j}^{a-1} X_j}{A_n^a};$$

where $\{(X_j; \mathcal{F}_j); j \geq 0\}$ are martingale differences that are identically distributed.

For simplicity, suppose that for some $\alpha \in (1; 2]$; $C > 0$ and all $j \geq 1$,

$$\mathbb{E} [|X_j| \mid \mathcal{F}_{j-1}] \leq C \text{ a.s.} \quad (15)$$

Theorem 5. Let $\{(X_j; \mathcal{F}_j); j \geq 0\}$ be identically distributed martingale differences satisfying (15). Let $p \geq 1$, and assume that

$$\left\{ \begin{array}{ll} E|X_1|^{\frac{p-1}{a+1}} < \infty & \text{if } 0 < a < 1 - \frac{1}{p}; \\ E|X_1|^p \log(e \vee |X_1|) < \infty & \text{if } a = 1 - \frac{1}{p}; \\ E|X_1|^p < \infty & \text{if } 1 - \frac{1}{p} < a \leq 1: \end{array} \right. \quad (16)$$

Then

$$\sum_{n=1}^{\infty} n^{p-2} P\left\{ \left| \sum_{j=0}^n A_{n-j}^{a-1} X_j \right| > A_n^a \right\} < \infty \text{ for all } a > 0: \quad (17)$$

Remark: in the independent case, the result is due to Gut (1993).

5. Proofs of main results

The proofs are based on some maximal inequalities for martingales.

A. Relation between

$$P(\max_{1 \leq j \leq n} |X_j| > \epsilon) \text{ and } P(\max_{1 \leq j \leq n} |S_j| > \epsilon)$$

for martingale differences (X_j) :

Lemma A Let $\{(X_j; \mathcal{F}_j); 1 \leq j \leq n\}$ be a finite sequence of martingale differences. Then for any $\epsilon > 0$; $\epsilon \in (1; 2]$; $q \geq 1$, and $L \in \mathbb{N}$,

$$\begin{aligned} P\{\max_{1 \leq j \leq n} |X_j| > 2\epsilon\} &\leq P\{\max_{1 \leq j \leq n} |S_j| > \epsilon\} \\ &\leq P\{\max_{1 \leq j \leq n} |X_j| > \frac{\epsilon}{4(L+1)}\} \\ &\quad + C \epsilon^{-\frac{q(L+1)}{q+L}} (E m_n^q(\epsilon))^{\frac{1+L}{q+L}}; \end{aligned} \quad (18)$$

where $C = C(\epsilon; q; L) > 0$ is a constant depending only on ϵ ; q and L ,

$$m_n(\epsilon) = \sum_{j=1}^n E[|X_j| | \mathcal{F}_{j-1}]:$$

B. Relation between

$$P\left(\max_{1 \leq j \leq n} X_j > \epsilon\right) \text{ and } \sum_{1 \leq j \leq n} P(X_j > \epsilon)$$

for adapted sequences (X_j) :

Lemma B Let $\{(X_j; \mathcal{F}_j); 1 \leq j \leq n\}$ be an adapted sequence of r.v. Then for $\epsilon > 0$; $\delta > 0$ and $q \geq 1$,

$$\begin{aligned} P\left\{\max_{1 \leq j \leq n} X_j > \epsilon\right\} &\leq \sum_{j=1}^n P\{X_j > \epsilon\} \\ &\leq (1 + \epsilon^{-\delta}) P\left\{\max_{1 \leq j \leq n} X_j > \epsilon\right\} + \epsilon^{-\delta} \mathbb{E} m_n^q(\epsilon); \end{aligned}$$

where $m_n(\epsilon) = \sum_{j=1}^n \mathbb{E}[|X_j| | \mathcal{F}_{j-1}]$.

C. Relation between

$$P\left(\max_{1 \leq j \leq n} |S_j| > \epsilon\right) \text{ and } P(|S_n| > \epsilon)$$

for martingale differences (X_j) :

Lemma C Let $\{(X_j; \mathcal{F}_j); 0 \leq j \leq n\}$ be a finite sequence of martingale differences. Then for $\epsilon > 0$; $\epsilon \in (1; 2]$ and $q \geq 1$,

$$P\left\{\max_{1 \leq j \leq n} |S_j| > \epsilon\right\} \leq 2P\left\{|S_n| > \frac{\epsilon}{2}\right\} + \epsilon^{-q} 2^{q(n+1)} C^q(\epsilon) E m_n^q(\epsilon);$$

where $m_n(\epsilon) = \sum_{j=1}^n E[|X_j| | \mathcal{F}_{j-1}]$,

$$C(\epsilon) = \left(18 \epsilon^{3/2} = (\epsilon - 1)^{1/2}\right).$$

Thank you!

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