

Moments of Traces for Circular β -ensembles

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This is joint work with Sho Matsumoto

April 5, 2010

- Moments for Haar Unitary Matrices (D.E. Thm)
- Background for Circular β -Ensembles
- Moments for Circular β -Ensembles
- Proofs by Jack Polynomials

1. Moments for Haar Unitary Matrices

- ▶ What is Haar-invariant unitary matrix Γ_n ?

Mathematically,

Γ_n : normalized Haar measure on $U(n)$: set of n by n unitary matrices.

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1) The matrix Q in QR (Gram-Schmidt) decomposition of Y

$$2) \Gamma_n \stackrel{d}{=} Y(Y^*Y)^{-1/2}$$

► Theorem (Diaconis and Evans: 2001)

(a) $a = (a_1, \dots, a_k)$, $b = (b_1, \dots, b_k)$ with $a_j, b_j \in \{0, 1, 2, \dots, g\}$.
 X_1, \dots, X_k : i.i.d. $\mathbb{C}N(0, 1)$. If $n \gg \sum_{j=1}^k j a_j$

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 X_1, \dots, X_k : i.i.d. $\mathbb{C}N(0, 1)$. If $n \gg \sum_{j=1}^k j a_j - \sum_{j=1}^k j b_j$,

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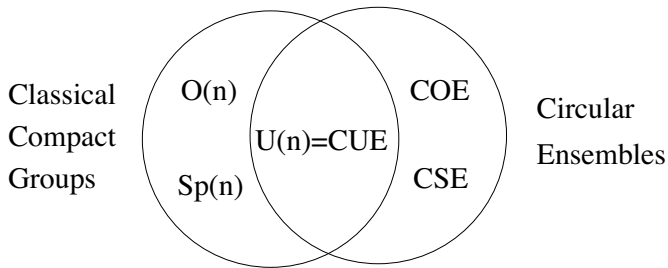
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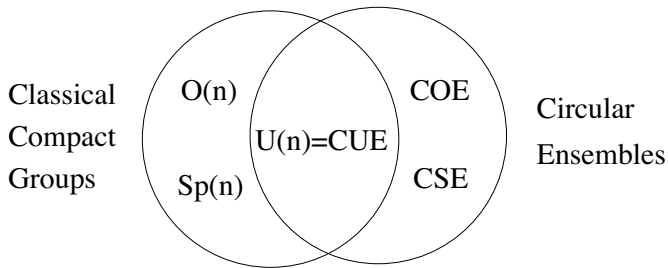
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(b) For j and k ,

$$\mathbb{E} [\text{Tr}(U_n^j) \overline{\text{Tr}(U_n^k)}] = \delta_{jk} j^{n/j}$$



Circular Ensembles and Haar-invariant Matrices from Classical Compact Groups



Circular Ensembles and Haar-invariant Matrices from Classical Compact Groups

Diaconis (2004) believes there is a good formula for COE and CSE

2. Background for Circular β -Ensembles

► Probability density function

$e^{i\theta_1}, \dots, e^{i\theta_n}$: eigenvalues of Haar-invariant unitary matrix.

pdf: $f(\theta_1, \dots, \theta_n | \beta = 2)$, where

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- This model: *circular β -ensemble* ($\beta = 1, 2, 4$) by physicist Dyson for study of nuclear scattering data

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Killip & Nenciu: Matrix models for circular ensembles

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- This suggest: moments for general β -ensemble depend on n

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$$\lambda = (3, 2, 2) : j\lambda_j = 7, m_2(\lambda) = 2, m_3(\lambda) = 1, l(\lambda) = 3,$$

$$p_\lambda = (\sum_i \lambda_i^3) (\sum_i \lambda_i^2)^2$$

$\alpha > 0, K \geq 1, n \geq 1$, define

$$A = \left(1 - \frac{j\alpha}{n} \frac{1j}{K + \alpha} \delta(\alpha \geq 1)\right)^K$$

$$B = \left(1 + \frac{j\alpha}{n} \frac{1j}{K + \alpha} \delta(\alpha < 1)\right)^K$$

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Let $\theta_1, \dots, \theta_n \stackrel{i.i.d.}{\sim} f(\theta), \alpha = 2/\beta$.

- $Z_n = (e^{i\theta_1}, \dots, e^{i\theta_n})$,
- $p_\mu(Z_n) = p_\mu(e^{i\theta_1}, \dots, e^{i\theta_n})$

Theorem

(a) If $n = K = j\mu$, then

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If $\mu \notin \nu$ and $n \in K = j\mu j \cup j\nu j$, then

$$\left| \mathbb{E}[p_\mu(Z_n) \overline{p_\nu(Z_n)}] \right| \leq \max\{j_A, j_B\} \alpha^{(l(\mu)+l(\nu))/2} (z_\mu z_\nu)^{1/2}$$

Theorem

(a) If $n \in K = j\mu j$, then

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If $\mu \notin \nu$ and $n \in K = j\mu j \cup j\nu j$, then

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(c) $\exists C = C(\beta)$ s.t. $\forall m \geq 1, n \geq 2$

$$\left| \mathbb{E}[jp_m(Z_n)f^2] \right| \leq n C \frac{n^3 2^{n\beta}}{m^{1 \wedge \beta}}$$

Take $\beta = 2$, then $A = B = 1$. We recover

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Corollary

$\delta\beta > 0$,

$$(a) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left[p_\mu(Z_n) \overline{p_\nu(Z_n)} \right] = \delta_{\mu\nu} \left(\frac{2}{\beta} \right)^{l(\mu)} z_\mu;$$

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$$(b) \quad \lim_{m \rightarrow \infty} \mathbb{E} [j p_m(Z_n) f^2] = n \quad \text{for any } n \geq 2.$$

Corollary

$\mu \neq \nu : K = j\mu j - j\nu j$. If $n \geq 2K$, then

$$(a) \quad \left| \frac{\mathbb{E}[j p_\mu(Z_n) j^2]}{\alpha^{l(\mu)} z_\mu} - 1 \right| \leq \frac{6j1 \alpha^{jK}}{n};$$

Corollary

$\mu \neq \nu : K = j_{\mu} j_{\nu}$. If $n \geq 2K$, then

$$(a) \quad \left| \frac{\mathbb{E}[j p_{\mu}(Z_n) j^2]}{\alpha^{l(\mu)} z_{\mu}} - 1 \right| \leq \frac{6j_{\nu}}{n} \alpha^{jK};$$

$$(b) \quad \left| \mathbb{E} \left[p_{\mu}(Z_n) \overline{p_{\nu}(Z_n)} \right] \right| \leq \frac{6j_{\nu}}{n} \alpha^{jK} \alpha^{(l(\mu)+l(\nu))/2} (z_{\mu} z_{\nu})^{1/2}.$$

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$$\mathbb{E}[jp_1(Z_n)f^2] = \frac{2}{\beta} \frac{n}{n(1 + 2\beta^{-1})} = \begin{cases} \frac{2n}{n+1}, & \text{if } \beta = 1 \\ 1, & \text{if } \beta = 2 \\ \frac{n}{2n-1}, & \text{if } \beta = 4 \end{cases}$$

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Exact formula is given next

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Orthogonal property: $Z_n = (e^{i\theta_1}, \dots, e^{i\theta_n})$

► Jack Polynomial

Jack polynomial $J_{\lambda}^{(\alpha)} = J_{\lambda}^{(\alpha)}(x_1$

Write

$$J_{\lambda}^{(\alpha)} = \sum_{\rho: |\rho|=|\lambda|} \theta_{\rho}^{\lambda}(\alpha) p_{\rho}$$
$$p_{\rho} = \sum_{\lambda: |\lambda|=|\rho|} \Theta_{\rho}^{\lambda}(\alpha) J_{\lambda}^{(\alpha)}$$

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For $j_{\mu} j_{\nu} = j_{\nu} j_{\mu} = K$,

$$\mathbb{E} \left[p_{\mu}(Z_n) \overline{p_{\nu}(Z_n)} \right] = \sum_{\lambda \vdash K: l(\lambda) \leq n} \Theta_{\mu}^{\lambda}(\alpha) \Theta_{\nu}^{\lambda}(\alpha) \mathbb{E} (J_{\lambda}^{(\alpha)} \overline{J_{\lambda}^{(\alpha)}})$$

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- relationship between $\theta_\rho^\lambda(\alpha)$ and $\Theta_\rho^\lambda(\alpha)$

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$$C_\lambda(\alpha) = \prod_{(i,j) \in \lambda} \left\{ (\alpha(\lambda_i - j) + \lambda'_j - i + 1)(\alpha(\lambda_i - j) + \lambda'_j - i + \alpha) \right\}$$

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$$N_\lambda^\alpha(n) = \prod_{(i,j) \in \lambda} \frac{n + (j - 1)\alpha}{n + j\alpha} \frac{(i - 1)}{i}$$

Young diagram

Main proof:

- play $C_\lambda(\alpha)$
- play $N_\lambda^\alpha(n)$
- use orthogonal relations of $\theta_\mu^\lambda(\alpha)$

► Examples

$$\mathbb{E}[jp_1(Z_n)j^4] = \frac{2n\alpha^2(n^2 + 2(\alpha - 1)n - \alpha)}{(n + \alpha - 1)(n + \alpha - 2)(n + 2\alpha - 1)}$$

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$$\begin{aligned}\mathbb{E}[jp_1(Z_n)J^4] &= \frac{2n\alpha^2(n^2 + 2(\alpha - 1)n - \alpha)}{(n + \alpha - 1)(n + \alpha - 2)(n + 2\alpha - 1)} \\ &= \begin{cases} \frac{8(n^2+2n-2)}{(n+1)(n+3)}, & \text{if } \beta = 1 \\ 2, & \text{if } \beta = 2 \\ \frac{2n^2-2n-1}{(2n-1)(2n-3)}, & \text{if } \beta = 4 \end{cases}\end{aligned}$$

$$\mathbb{E} \left[p_2(Z_n) \overline{p_1(Z_n)^2} \right]$$

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$$\begin{aligned}
& \mathbb{E} \left[p_2(Z_n) \overline{p_1(Z_n)^2} \right] \\
&= \frac{2\alpha^2(\alpha - 1)n}{(n + \alpha - 1)(n + 2\alpha - 1)(n + \alpha - 2)} \\
&= \begin{cases} \frac{8}{(n+1)(n+3)}, & \text{if } \beta = 1 \\ 0, & \text{if } \beta = 2 \\ \frac{-1}{(2n-1)(2n-3)}, & \text{if } \beta = 4 \end{cases}
\end{aligned}$$

The End!

Thanks for your patience!