

# Optimal Risk Probability for First Passage Models

| in Semi-Markov Decision Processes

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# Outline

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# 1. Motivation

**Background:** Reliability engineering, and risk analysis

**Problem:**  $\sup_{\pi} P_i^{\pi}(\tau_B > \epsilon)$ ,

$\tau_B$  is an initial state

$\pi$  is a policy

$B$  is a given target set

$\tau_B$  is a first passage time to  $B$

$\epsilon$  is a threshold value.

## 2. Semi-Markov Decision Processes

The model of SMDP:

$$fS; B; (A(i); i \in S); Q(t; j|i; a)g$$

where

$\in S$  : the state space, a denumerable set;

$\in B$ : a given target set, a subset of  $S$ ;

$\in A(i)$  : finite set of actions available at  $i \in S$ ;

$\in Q(t; j|i; a)$  : semi-Markov kernel,  $a \in A(i); i, j \in S$ ;

## Notation:

$\pi$  **Policy**  $\pi$ : A sequence  $\pi = \{f_n; n = 0, 1, \dots, g\}$  of stochastic kernels  $f_n$  on the action space  $A$  given  $H_n$  satisfying

$$f_n(A(i_n) | (0; i_0; \dots; 0; a_0; \dots; t_{n-1}; i_{n-1}; \dots; a_{n-1}; t_n; i_n)) = 1$$

$\pi$  **Stationary policy**: measurable  $f, f(i; \dots) \in A(i)$  for all  $(i; \dots)$

$\pi$   $P_{(i; \dots)}^\pi$ : Probability measure on  $(S \times [0; 1]) \times (\prod_{i \in S} A(i))$

$\pi$   $S_n; J_n; A_n$ :  $n$ -th decision epoch, the state and action at the  $S_n$ , respectively.

**Assumption A.** There exist  $\epsilon > 0$  and  $\delta > 0$  such that

$$\sum_{j \in S} Q(\pm; j | i; a) \cdot \mathbf{1}_{|j-i| \leq \delta}; \text{ for all } (i; a) \in K:$$

Assumption A )  $P_{(i; s)}^{\frac{1}{4}}(fS_1 = \mathbf{1} g) = 1$

**Semi-Markov decision process**  $f(Z(t); A(t); t, 0)g$ :

$$Z(t) = J_n; A(t) = A_n; \text{ for } S_n \leq t < S_{n+1}$$

The first passage time into  $B$ , is defined by

$$\tau_B := \inf \{ t \geq 0 \mid Z(t) \in B \}; \text{ (with } \inf \emptyset := \infty \text{);}$$

### 3. Optimality Problems

The risk probability:

$$F^{\frac{1}{4}}(i; s) := P_{(i; s)}^{\frac{1}{4}}(\mathcal{C}_B \cdot s)$$

The optimal value:

$$F_{\alpha}(i; s) := \inf_{\frac{1}{4}2\Pi} F^{\frac{1}{4}}(i; s);$$

**Definition 1.** A policy  $\frac{1}{4}^{\alpha} 2 \mid$  is called optimal if

$$F^{\frac{1}{4}^{\alpha}}(i; s) = F_{\alpha}(i; s) \quad \forall (i; s) \in S \in R:$$

<sup>2</sup> Existence and computation of optimal policies ???

## 4. Optimality Equation

For  $i \in B^c$ ;  $a \in A(i)$ , and  $s \geq 0$ , let

$$T^a u(i; s) := Q(s; Bji; a) + \sum_{j \in B^c} \int_0^s Q(dt; jji; a) u(j; s - t);$$

with  $u \in F_{[0,1]}$  (the set of measurable functions  $0 \leq u \leq 1$ ),

$$Q(s; Bji; a) := \sum_{j \in B} Q(s; jji; a); \quad T^a u(i; s) := 0 \text{ for } s < 0;$$

Then, define operators  $T$  and  $T^f$ :

$$Tu(i; s) := \min_{a \in A(i)} T^a u(i; s); \quad T^f u(i; s) := T^{f(i; s)} u(i; s);$$

for each stationary policy  $f$ .



**Theorem 1.** Let Under Assumption A, we have

(a)  $F^f = \lim_{n \rightarrow \infty} u_n^f$ , where  $u_n^f := T^f u_{n-1}^f; u_{-1}^f := 1;$

(b)  $F^f$  satisfied the equation,  $u = T^f u$ , for all  $f \in F;$

**2** Theorem 1 gives an approximation of risk probability  $F^f.$

For each  $(i; s) \in B^c \in R_+$  and  $\frac{1}{4} \in \mathbb{I}$ , let

$$F_{i-1}^{\frac{1}{4}}(i; s) := 1;$$

$$F_n^{\frac{1}{4}}(i; s) := 1 - \sum_{m=0}^n P_{(i; s)}^{\frac{1}{4}}(S_m \cdot s < S_{m+1}; J_k \in B^c; 0 \leq k \leq m)$$

**Theorem 2.** Let  $F_n^\alpha(i; s) := \inf_{\frac{1}{4}} F_n^{\frac{1}{4}}(i; s)$ , then

(a)  $F_{n+1}^\alpha = TF_n^\alpha$  for all  $n \geq j \geq 1$ , and  $\lim_{n \rightarrow \infty} F_n^\alpha = F_\alpha$ .

(b)  $F_\alpha$  satisfies the **optimality equation**:  $F_\alpha = TF_\alpha$ .

(c)  $F_\alpha$  is the maximal fixed point of  $T$  in  $F_{[0;1]}$ .

**Remark 1.**

$\Rightarrow$  Theorem 2(a) gives a **value iteration algorithm** for computing the optimal value function  $F_\alpha$ .

$\Rightarrow$  Theorem 2(b) establishes the **optimality equation**.

## 5. Existence of Optimality Policies

To ensure the existence of optimal policies, we introduce the following condition.

**Assumption B.** For every  $(i, s) \in B^c \times R$  and  $f$ ,

$$P_{(i,s)}^f(\zeta_B < 1) = 1:$$

To verify Assumption B, we have a fact below:

**Theorem 3.** If there exists a constant  $\delta > 0$  such that

$$\sum_{j \in B} Q(1; j, i; a) \geq \delta \text{ for all } i \in B^c; a \in R$$

**Theorem 4.** Under Assumptions A and B, we have

- (a)  $F^f$  and  $F_\alpha$  are the unique solution in  $F_{[0;1]}$  to equations  $u = T^f u$  and  $u = T u$ , respectively;
- (b) any  $f$ , such that  $F_\alpha = T^f F_\alpha$ , is optimal;
- (c) there exists a stationary policy  $f^\alpha$  satisfying the optimality equation:  $F_\alpha = T F_\alpha = T^{f^\alpha} F_\alpha$ ; and such policy  $f^\alpha$  is optimal.

**Remark 2.**

<sup>2</sup> Theorem 4(c) shows the existence of an optimal policy.

To give the existence of special optimal policies, let

$$A^\alpha(i; \delta) := \{f \in \mathcal{F} : \sum_{j \in B^c} f_{ij} F^\alpha(j; \delta) = T^\alpha F^\alpha(i; \delta)g\}$$

$$A^\alpha(i) := \bigcap_{\delta > 0} A^\alpha(i; \delta)$$

**Theorem 5.** If  $\sup_{i \in B^c} \sup_{a \in A(i)} Q(t; B^c | i; a) < 1$  for some  $t > 0$ , and Assumptions A and B hold, then,

- (a) for any  $g \in G := \{fg | g(i) \in A(i) \forall i \in S\}$ ,  $F^g$  is the unique solution in  $F_{[0,1]}$  to the equation:  $u = T^g u$ ;
- (b) there exists an optimal policy  $f \in G$  if and only if  $A^\alpha(i) \neq \emptyset$ ; for all  $i \in B^c$ .

## 5. Numerable examples

**Example 5.1.** Let  $S = \{1, 2, 3\}$ ,  $B = \{3\}$ , where

state 1: the good state

state 2: the medium state

state 3: the failure state

Let  $A(1) = \{a_{11}, a_{12}\}$ ;  $A(2) = \{a_{21}, a_{22}\}$ ;  $A(3) = \{a_{31}\}$ .

The semi-Markov kernel is of the form:

$$Q(t; j | i; a) = H(t | i; a) p(j | i; a)$$

$H(t; j, i; a)$  : the distribution functions of the sojourn time

$p(j; j, i; a)$ : the transition probabilities.

$$H(t; j=1; a_{11}) := \begin{cases} 1 - e^{-0.25t}; & t \in [0; 25]; \\ 1; & t > 25; \end{cases}$$

$$H(t; j=2; a_{21}) := \begin{cases} 1 - e^{-0.2t}; & t \in [0; 20]; \\ 1; & t > 20; \end{cases}$$

$$H(t; j=3; a_{31}) := 1 - e^{-0.2t};$$

$$H(t; j=1; a_{12}) = 1 - e^{-0.08t};$$

$$H(t; j=2; a_{22}) = 1 - e^{-0.15t};$$

$$\begin{aligned}
p(1 \ j \ 1; a_{11}) &= 0; & p(2 \ j \ 1; a_{11}) &= \frac{9}{20}; & p(3 \ j \ 1; a_{11}) &= \frac{11}{20}; \\
p(1 \ j \ 1; a_{12}) &= 0; & p(2 \ j \ 1; a_{12}) &= \frac{1}{2}; & p(3 \ j \ 1; a_{12}) &= \frac{1}{2}; \\
p(1 \ j \ 2; a_{21}) &= \frac{1}{5}; & p(2 \ j \ 2; a_{21}) &= 0; & p(3 \ j \ 2; a_{21}) &= \frac{4}{5}; \\
p(1 \ j \ 2; a_{22}) &= \frac{1}{4}; & p(2 \ j \ 2; a_{22}) &= 0; & p(3 \ j \ 2; a_{22}) &= \frac{3}{4}; \\
p(3 \ j \ 3; a_{31}) &= 1;
\end{aligned}$$

Using the **value iteration algorithm** in Theorem 2, we obtain some computational results as in Figure 1 and Figure 2.



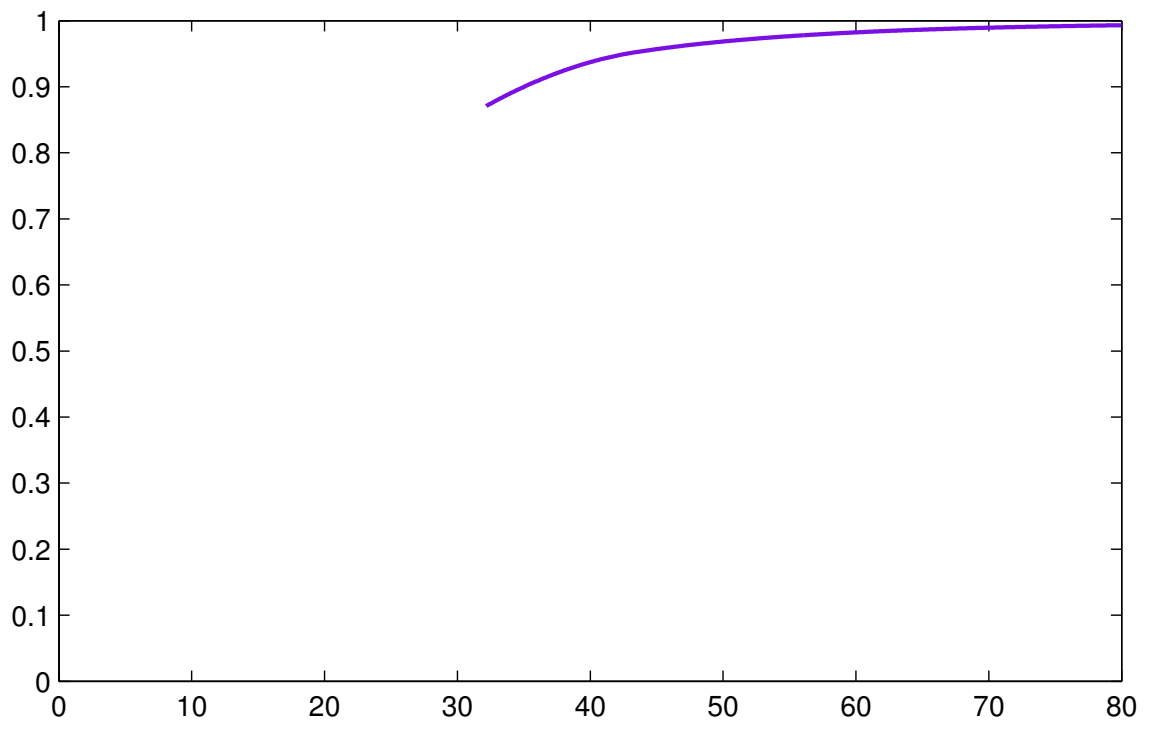


Figure 1. The functions  $T^a F^a(i; s)$

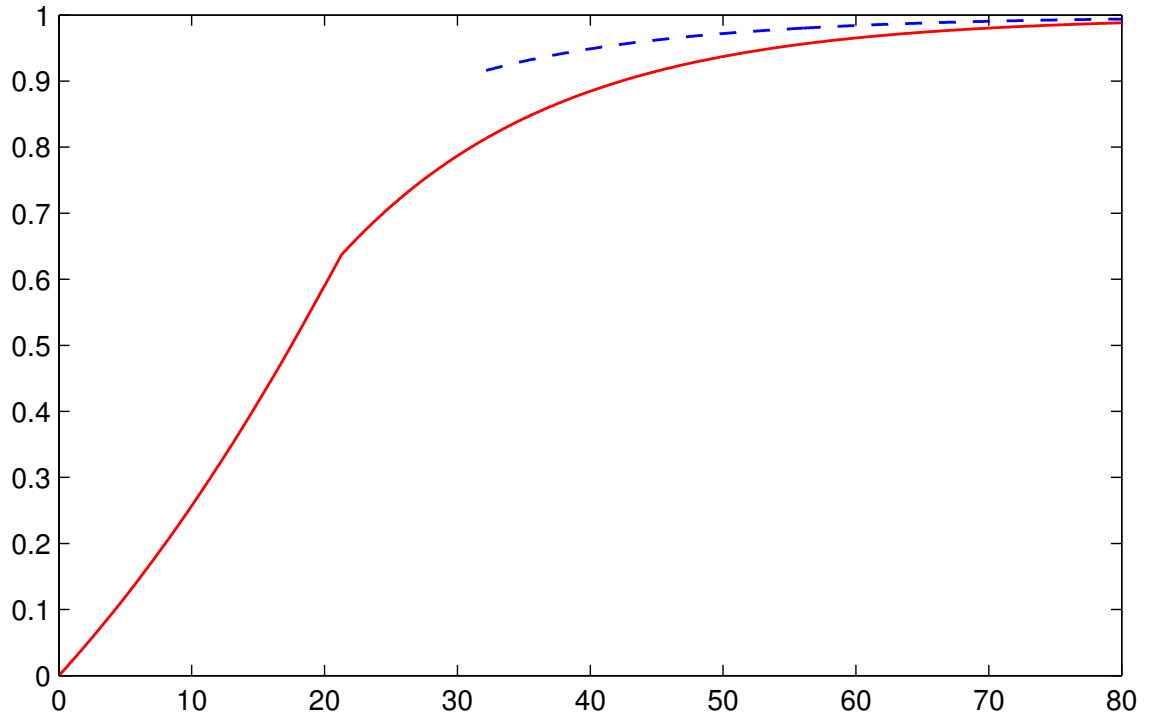


Figure 2. The value function  $F^a(i; s)$

More clearly, we have

$$F^{\alpha}(1; \varsigma) = \begin{cases} T^{a_{11}} F^{\alpha}(1; \varsigma); & 0 \cdot \varsigma < 21:36; \\ T^{a_{11}} F^{\alpha}(1; \varsigma) = T^{a_{12}} F^{\alpha}(1; \varsigma); & \varsigma = 21:36; \\ T^{a_{12}} F^{\alpha}(1; \varsigma); & 21:36 < \varsigma < 29:3; \\ T^{a_{11}} F^{\alpha}(1; \varsigma) = T^{a_{12}} F^{\alpha}(1; \varsigma); & \varsigma = 29:3; \\ T^{a_{11}} F^{\alpha}(1; \varsigma) (= 0:7742); & \varsigma > 29:3; \end{cases}$$

$$F^{\alpha}(2; \varsigma) = \begin{cases} T^{a_{21}} F^{\alpha}(2; \varsigma); & 0 \cdot \varsigma < 18:54; \\ T^{a_{21}} F^{\alpha}(2; \varsigma) = T^{a_{22}} F^{\alpha}(2; \varsigma); & \varsigma = 18:54; \\ T^{a_{22}} F^{\alpha}(2; \varsigma); & 18:54 < \varsigma < 23:82; \\ T^{a_{21}} F^{\alpha}(2; \varsigma) = T^{a_{22}} F^{\alpha}(2; \varsigma); & \varsigma = 23:82; \\ T^{a_{21}} F^{\alpha}(2; \varsigma) (= 0:8542); & \varsigma > 23:82; \end{cases}$$

Define a policy  $f^\alpha$  by

$$f^\alpha(1; s) = \begin{cases} a_{11}; & 0 \leq s \leq 21.36; \\ a_{12}; & 21.36 < s \leq 29.3; \\ a_{11}; & s > 29.3; \end{cases}$$

$$f^\alpha(2; s) = \begin{cases} a_{21}; & 0 \leq s \leq 18.54; \\ a_{22}; & 18.54 < s \leq 23.82; \\ a_{21}; & s > 23.82; \end{cases}$$

Then, we have

$${}^2 F^\alpha(i; s) = T^{f^\alpha} F^\alpha(i; s) \text{ for } i = 1, 2 \text{ and all } s \geq 0,$$

${}^2 f^\alpha$  is an optimal stationary policy.

$$A^{\alpha}(1; \varsigma) = \begin{cases} fa_{11}g; & 0 \cdot \varsigma < 21:36; \\ fa_{11}; a_{12}g; & \varsigma = 21:36; \\ fa_{12}g; & 21:36 < \varsigma < 29:3; \\ fa_{11}; a_{12}g; & \varsigma = 29:3; \\ fa_{11}g; & \varsigma > 29:3; \end{cases}$$

$$A^{\alpha}(2; \varsigma) = \begin{cases} fa_{21}g; & 0 \cdot \varsigma < 18:54; \\ fa_{21}; a_{22}g; & \varsigma = 18:54; \\ fa_{22}g; & 18:54 < \varsigma < 23:82; \\ fa_{21}; a_{22}g; & \varsigma = 23:82; \\ fa_{21}g; & \varsigma > 23:82; \end{cases}$$

Hence,

$$A^\alpha(1) = \bigcap_{\delta > 0} A^\alpha(1; \delta) = \dots; A^\alpha(2) = \bigcap_{\delta > 0} A^\alpha(2; \delta) = \dots$$

which show there is no optimal policy in  $G$ .

**Remark 3.** This shows that the assumption in the previous literature is not satisfied for this example !!!

**Example 5.2.** Let  $S = f1;2g$ ,  $B = f2g$ ;

$$A(1) = fa_{11};a_{12}g; A(2) = fa_{21}g;$$

$Q(t;j j i; a)$  is given by

$$Q(t;j j 1; a_{11}) = \begin{cases} 1=2; & \text{if } t_s 1;j = 1;2; \\ 0; & \text{otherwise;} \end{cases}$$

$$Q(t;j j 1; a_{12}) = \begin{cases} 1; & \text{if } t_s 2;j = 2; \\ 0; & \text{otherwise;} \end{cases}$$

$$Q(t;j j 2; a_{21}) = \begin{cases} 1 j e^i t; & \text{if } t_s 0;j = 2; \\ 0; & \text{otherwise;} \end{cases}$$

Assumptions A and B holds in this example.

We now define a policy  $d$  as follows:

$$d(1; s) = \begin{cases} a_{12}; & 0 \leq s < 2; \\ a_{11}; & s \geq 2; \end{cases}$$

Then, by Theorem 1, we have  $F^d(1; s) = \lim_{n \rightarrow \infty} F_n^d(1; s)$ , which yields

$$F^d(1; s) = \begin{cases} 0; & 0 \leq s < 2; \\ 1; & s = 2; \\ 1=2; & 2 < s < 3; \end{cases}$$

Hence,  $F^d(1; s)$  is not a distribution function of  $s$ .



**Many Thanks !!!**