

# Absolute purity in motivic homotopy theory

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joint work with F. Déglise, J. Fasel and A. Khan

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# The absolute purity conjecture

Grothendieck's **absolute (cohomological) purity conjecture** (SGA5, Expose I 3.1.4) is the following statement: if  $i : Z \rightarrow X$  is a closed immersion between noetherian regular schemes of pure codimension  $c$ ,  $n \in \mathcal{O}(X)$  and  $i^* = \mathbb{Z} = n\mathbb{Z}$ , then the étale cohomology sheaf supported in  $Z$  with values in  $\mathbb{Z}$  can be computed as

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This conjecture has been solved by Gabber.

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Based on Thomason's method + rigidity for algebraic  $K$ -theory

## Importance of the absolute purity conjecture

The absolute purity property, together with resolution of singularities, is frequently used in cohomological studies of schemes:

**Grothendieck's absolute purity conjecture**

Motivic homotopy theory

The fundamental class

Absolute purity in motivic homotopy theory

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- Construct Gysin morphisms and establish intersection theory.
- Study the coniveau spectral sequence.

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- Our work: study absolute purity in the framework of motivic homotopy theory.
- Main result: the absolute purity in motivic homotopy theory is satisfied with rational coefficients in mixed characteristic.

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**Motivic homotopy theory**  
The fundamental class

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# Motivic homotopy theory

- The motivic homotopy theory or  $\mathbb{A}^1$ -homotopy theory is introduced by Morel and Voevodsky (1998) as a framework to study cohomology theories in algebraic geometry, by importing tools from algebraic topology
- Idea: use the affine line  $\mathbb{A}^1$  as a substitute of the unit interval to get an algebraic version of the homotopy theory
- Can be used to study cohomology theories such as algebraic  $K$ -theory, Chow groups (motivic cohomology) and many others
- Advantage: has many a lot of structures coming from both topological and algebraic geometrical sides

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- Non-commutative geometry and singularity categories (Tabuada, Blanc-Robalo-Toën-Vezzosi)

## Some topological background

- A **spectrum**  $\mathbb{E}$  is a sequence  $(E_n)_{n \in \mathbb{N}}$  of pointed spaces (e.g. CW-complexes or simplicial sets) together with continuous maps  $\sigma_n : S^1 \wedge E_n \rightarrow E_{n+1}$  called **suspension maps**



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- The theory stems from the Freudenthal suspension theorem: if  $E_i = X \wedge S^i$  for some  $X \in \mathit{Top}$  (i.e.  $E$  is the **suspension spectrum** of  $X$ ), then the sequence  $i \mapsto \pi_{n+i}(E_i)$  stabilizes

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- From an  $\infty$ -categorical point of view, the category of spectra is the stabilization of the category of spaces, and is the universal stable (triangulated) category

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- Bigraded  $\mathbb{A}^1$ -**homotopy sheaves**: for  $X \in \mathbf{H}(S)$ ,  $\mathbb{A}_{a,b}^1(X)$  is the Nisnevich sheaf on  $Sm_S$  associated to the presheaf

$$U \mapsto [U \wedge S^{a-b} \wedge \mathbb{G}_m^b; X]_{\mathbf{H}_\bullet(S)}$$

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- $\mathbf{SH}(S)$  is the universal stable  $\infty$ -category which satisfies Nisnevich descent and  $\mathbb{A}^1$ -invariance (Robalo, Drew-Gallauer)

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## Motivic spectra

Every object in  $\mathbf{SH}(S)$  represents a bigraded cohomology theory

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- Milnor-Witt spectrum  $\mathbf{H}_{MW}\mathbb{Z}$  represents Milnor-Witt motivic cohomology/higher Chow-Witt groups (Deglise-Fasel)



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- In particular,  $End(\mathbb{1}_k)_{\mathbf{SH}(k)} \simeq GW(k)$  is the Grothendieck-Witt groups of symmetric bilinear forms over  $k$

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- The 1-line is also computed (Rondigs-Spitzweck-Østvær):

$$0 \rightarrow K_2^M \rightarrow {}_{n+1,n}(\mathbb{1}_k) \rightarrow {}_{n+1,n}f_0(\mathbf{KQ})$$

## The six functors formalism

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- They satisfy formal properties axiomatizing important theorems such as duality, base change and localization.

## Thom spaces and relative purity

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Grothendieck's absolute purity conjecture  
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- In the presence of an *orientation*, we recover the usual relative purity

## Orientations

- An **absolute motivic spectrum** is the data of  $\mathbb{E}_X \in \mathbf{SH}(X)$  for every scheme  $X$ , together with natural isomorphisms  $f^* \mathbb{E}_X \simeq \mathbb{E}_Y$  for every morphism  $f : Y \rightarrow X$



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- A theory of *fundamental classes* aims at establishing a cohomological intersection theory
- For oriented spectra, Deglise defined fundamental classes using Chern classes

## Bivariant groups

- For  $f : X \rightarrow S$  be a separated morphism of finite type,  $v \in K_0(X)$  and  $\mathbb{E} \in \mathbf{SH}(S)$ , define the  **$\mathbb{E}$ -bivariant groups** (or **Borel-Moore  $\mathbb{E}$ -homology**) as

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- If  $S$  is a field and  $\mathbb{E} = \mathbf{HZ}$ , then  $\mathbb{E}_i(X=S; v) = CH_r(X; i)$  are the higher Chow groups, where  $r$  is the virtual rank of  $v$



## Functoriality of bivariant groups

- Base change:

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ g \downarrow & \Delta & \downarrow f \\ T & \xrightarrow{p} & S \end{array}$$

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- Product: if  $\mathbb{E}$  has a ring structure,  $X \xrightarrow{f} Y \xrightarrow{g} S$

$$\mathbb{E}_m(X=Y; w) \otimes \mathbb{E}_n(Y=S; \nu) \rightarrow \mathbb{E}_{m+n}(X=S; w + f \nu)$$

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all compatible with compositions
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- The construction uses the *deformation to the normal cone*

## Euler class and excess intersection formula

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- **Motivic Gauss-Bonnet formula** (Levine, Deglise-J.-Khan)  
 For  $p : X \rightarrow S$  a smooth and proper morphism

$$(X=S) = p_* e(T_p)$$

where  $(X=S)$  is the *categorical Euler characteristic*

## The absolute purity property

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- From this property Cisinski-Déglise deduce that the rational motivic Eilenberg-Mac Lane spectrum  $\mathbf{H}\mathbb{Q}$  also satisfies absolute purity, mainly because  $\mathbf{H}\mathbb{Q}$  is a direct summand of  $\mathbf{KGL}_{\mathbb{Q}}$  by the Grothendieck-Riemann-Roch theorem

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**Theorem** (Deglise-Fasel-J.-Khan):

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- Therefore it suffices to show that the minus part satisfies absolute purity

## The first proof

- By a devissage theorem of Schlichting and an argument similar to the case of **KGL**, one can show that the Hermitian K-theory spectrum **KQ** satisfies absolute purity

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- This proves the absolute purity of  $\mathbb{1}_{\mathbb{Q}}$  when 2 is invertible on the base scheme, since  $\mathbf{KQ}$  is only well-defined in this case

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- The key lemma then reduces the absolute purity of  $\mathbb{1}_{\mathbb{Q}}$  in mixed characteristic to the case of  $\mathbb{Q}$ -schemes, which can be proved using Popescu's theorem: a closed immersion of a finite regular schemes over a perfect field is a limit of closed immersions of smooth schemes

## Some applications

- Our method can be used to deduce the following new results in mixed characteristic:
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- The rational bivariant groups  $H_0^{\mathbb{A}^1}(X=S; \nu)_{\mathbb{Q}}$  agree with the rational Chow-Witt groups, and can be computed by the Gersten complex associated to Milnor-Witt K-theory

## Some applications

- Our method can be used to deduce the following new results in mixed characteristic:
  - The six functors preserve constructible objects in the rational stable motivic homotopy category  $\mathbf{SH}(\cdot; \mathbb{Q})$
  - The Grothendieck-Verdier duality holds for  $\mathbf{SH}(\cdot; \mathbb{Q})$
  - The homotopy  $t$ -structure on  $\mathbf{SH}(\cdot; \mathbb{Q})$  behaves as expected
- The rational stable motivic homotopy category has a (unique)  $SL$ -orientation, that is, the Thom space of a vector bundle only depends on its rank and its determinant
- The rational bivariant groups  $H_0^{\mathbb{A}^1}(X=S; \nu)_{\mathbb{Q}}$  agree with the rational Chow-Witt groups, and can be computed by the Gersten complex associated to Milnor-Witt K-theory
- Related work: absolute purity of the sphere spectrum over a Dedekind domain (Frankland-Nguyen-Spitzweck, work in progress)

Thank you!