Projective Bundle Theorem in MW-Motives

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October 9, 2020

Motivation

Suppose $0 \quad i \quad n$, we have:

$$H^i(\mathrm{RP}^n, \mathrm{Z}) = \begin{cases} \mathrm{Z} & \text{if } i = 0 \text{ or } i = n \text{ and } n \text{ is odd} \\ \mathrm{Z}/2\mathrm{Z} & \text{if } i > 0 \text{ is even} \\ 0 & \text{else.} \end{cases}$$

Theorem (Fasel, 2013)

$$\widetilde{CH}^i(\mathrm{P}^n) = \begin{cases} GW(k) & \textit{if } i = 0 \textit{ or } i = n \textit{ and } n \textit{ is odd} \\ \mathrm{Z} & \textit{if } i > 0 \textit{ is even} \\ 2\mathrm{Z} & \textit{else} \end{cases}$$

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Question

- A motivic explanation?
- How about projective bundles?

Chow Groups

- $CH^n(X) = Z f$cycles of codimension $ng/$rational equivalence:

$$
\begin{array}{ccccc}
\bigoplus_{y \in X^{(n \;\; 1)}} k(y)^* & \xrightarrow{div} & \bigoplus_{y \in X^{(n)}} \mathrm{Z} & \longrightarrow & 0\,. \\
 & & \vdots\, H & & \\
 & & CH^n(X) & &
\end{array}
$$

- Projective bundle theorem:

$$CH^n(\mathrm{P}(E)) = \bigoplus_{i=0}^{rk(E)-1} CH^{n-i}(X) \quad \mathrm{P}(E) = \bigoplus_{i=0}^{rk(E)-1} X(i)[2i].$$

- Chern class:

$$c_i(E)\ 2\ CH^i(X).$$

Chow-Witt Groups

Suppose X is smooth and L 2 $Pic(X)$. We have the Gersten complex:

$$\bigoplus_{y \in X^{(n-1)}} \mathbf{K}_1^{MW}(k(y), L \quad {}^*_y) \xrightarrow{div} \bigoplus_{y \in X^{(n)}} \mathbf{GW}(k(y), L \quad {}^*_y) \xrightarrow{div} \bigoplus_{y \in X^{(n+1)}} \mathbf{W}(k(y), L \quad {}^*_y) .$$

$$\bigoplus_{y \in X^{(n)}} \mathbf{GW}(k(y), L \quad {}^*_y) \overset{H}{\dashrightarrow} \widetilde{CH}^n(X, L)$$

Chow-Witt Groups

- Suppose X is celluar. We have a Cartesian square:

$$
\begin{array}{ccccc}
\widetilde{CH}^n(X) & \longrightarrow & ker(\partial) & & CH^n(X) \longrightarrow H^{2n}(X(\mathrm{C}), \mathrm{Z}) \\
\downarrow & & & & \downarrow \qquad \searrow^{@} \\
H^n(X, \mathbf{I}^n) & \longrightarrow & & CH^n(X)/2 & \longrightarrow H^{n+1}(X, \mathbf{I}^{n+1}) \\
\downarrow & & & & \\
H^n(X(\mathrm{R}), \mathrm{Z}) & & & &
\end{array}
.
$$

- Pontryagin class
- Bockstein image of Stiefel-Whitney classes
- Orientation class

Four Motivic Theories

- Suppose $\mathbf{K} = MW, M, W, M/2$. We have a homotopy Cartesian:

$$\begin{array}{ccc} MW & \longrightarrow & W \\ \downarrow & & \downarrow \\ M & \longrightarrow & M/2 \end{array} .$$

Definition

Define the category of effective $\mathbf{K}$-motives over S with coefficients in R:

$$DM^{e}_{\mathbf{K}} = D[(X \times \mathbb{A}^1 \to X)^{-1}]$$

where D is the derived category of Nisnevich sheaves with $\mathbf{K}$-transfers.

- $\mathbf{K} = MW \Rightarrow$ Milnor-Witt Motives
- $\mathbf{K} = M \Rightarrow$ Voevodsky's Motives

Four Motivic Theories

Theorem (BCDFØ, 2020)

For any X 2 Sm/S and n 2 N, we have

$$[X, \mathrm{Z}(n)[2n]]_{\mathbf{K}} = \widetilde{CH}^n(X), CH^n(X), CH^n(X)/2$$

if $\mathbf{K} = MW, M, M/2$.

Theorem (Cancellation, BCDFØ, 2020)

Suppose $S = pt$. *For any* A, B 2 $DM^e_{\mathbf{K}}$, *we have*

$$[A, B]_{\mathbf{K}} \overset{\otimes(1)}{\underset{\cong}{!}} [A(1), B(1)]_{\mathbf{K}}.$$

Basic Calculations

- $A^n = Z$.
- $G_m = Z \quad Z(1)[1]$.
- $A^n\, n\, 0 = Z \quad Z(n)[2n \quad 1]$.
- $P^1 = Z \quad Z(1)[2]$.
- $A^n/(A^n\, n\, 0) = P^n/(P^n\, n\, pt) = Z(n)[2n]$.
- $E = X$ for any A^n-bundle E over X.

Hopf Map η

Definition

The multiplication map $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ induces a morphism

$$\mathbb{G}_m \wedge \mathbb{G}_m \to \mathbb{G}_m.$$

It's the suspension of a (unique) morphism $\eta \in [\mathbb{G}_m, \mathbb{1}]$, which is called the Hopf map.
It's also equal, up to a suspension, to the morphism

$$\begin{array}{ccc} \mathbb{A}^2 \setminus 0 & \to & \mathbb{P}^1 \\ (x, y) & \mapsto & [x : y] \end{array}.$$

Remark

The $\eta = 0$ if $K = M, M/2$, but never zero if $K = MW, W$!

$$\pi_3(S^2) = \mathbb{Z} \quad \text{Hopf}$$

MW-Motive of $\mathbb{P}^n$

Theorem (Y)

Suppose $n \in \mathbb{N}$ and $p : \mathbb{P}^n \to pt$.

1. *If n is odd, there is an isomorphism*

$$\mathbb{P}^n \xrightarrow{(p, c_n^{2i-1}, th_{n+1})} R \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} cone(\eta)(2i-1)[4i-2] \oplus R(n)[2n].$$

2. *If n is even, there is an isomorphism*

$$\mathbb{P}^n \xrightarrow{(p, c_n^{2i-1})} R \oplus \bigoplus_{i=1}^{\frac{n}{2}} cone(\eta)(2i-1)[4i-2].$$

Here $th_{n+1} = i_*(1)$ for some rational point $i : pt \to \mathbb{P}^n$.

$c_n^{2i-1} : \mathbb{P}^n \to cone(\eta)(2i-1)[4i-2]$

We have $cone(\eta) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$ in DM^e_M since $\eta = 0$. This implies

$$[\mathbb{P}^n, cone(\eta)(j)[2j]]_M = CH^j(\mathbb{P}^n) \oplus CH^{j+1}(\mathbb{P}^n).$$

We have an adjunction $\gamma^* : DM^e_{MW} \rightleftarrows DM^e_M : \gamma_*$.

Theorem (Y)

Suppose $j = 2i - 1 \leq n - 1$. The morphism

$$\begin{aligned} \gamma^* : \quad [\mathbb{P}^n, cone(\eta)(j)[2j]]_{MW} &\to [\mathbb{P}^n, cone(\eta)(j)[2j]]_M \\ c_n^j &\mapsto (c_1(O(1))^k, c_1(O(1))^{k+1}) \end{aligned}$$

is injective with $coker(\gamma^*) = \mathbb{Z}/2\mathbb{Z}$.

Splitness in MW-Motives

Definition

We say X 2 Sm/k splits in DM^{e}_{MW} if it's isomorphic to the form

Goal

Suppose E is a vector bundle. Find out the global definition of c_n^{2i-1} and th_{n+1} on $\mathrm{P}(E)$.

Motivic Stable Homotopy Category $SH(k)$

- $f\mathbb{P}^1$ spectra of simp. Nis. sheaves$g/$stable $\mathbb{A}^1$-equivalences.
- E-cohomologies:

$$[\Sigma^\infty X_+, E(q)[p]]_{\mathcal{SH}(k)} = E^{p;q}(X).$$

- $H^n(X, \mathbf{K}_n) = H_{\mathbf{K}}^{2n;n}(X) = CH^n(X), \widetilde{CH}^n(X),$,
  if $E = H\ \mathbb{Z}, H\widetilde{\mathbb{Z}},$.
- $(DM_{MW})_{\mathbb{Q}} = SH_{\mathbb{Q}}$.

Motivic Cohomology Spectra

Definition

Every motivic theory corresponds to a spectrum in $SH(k)$, namely

$$\begin{array}{cccc} H\widetilde{Z} & H\ Z & H_W Z & H\ Z/2\,. \\ \downarrow & \downarrow & \downarrow & \downarrow \\ MW & M & W & M/2 \end{array}$$

The spectrum represents the $cone(\eta)$ (induces the same cohomologies) of, for example, MW-motive is denoted by $H\widetilde{Z}/\eta$.

$H\widetilde{\mathbb{Z}}/\eta$

Theorem (Y)

We have a distinguished triangle

$$\mathbb{P}^1 \wedge H \ \mathbb{Z} \ \to \ H\widetilde{\mathbb{Z}}/\eta \ \to \ H \ \mathbb{Z} \quad H \ \mathbb{Z}/2[2] \ \to \ \mathbb{P}^1 \wedge H \ \mathbb{Z}[1].$$

Remark

The triangle doesn't split since applying $\pi_2()_0$ we get an exact sequence of Nisnevich sheaves

$$0 \ \to \ \mathbb{Z}/2\mathbb{Z} \ \to \ \mathcal{O}^* \ \to \ 2\mathcal{O}^* \ \to \ 0.$$

$\eta^i_{MW}(X)$

Definition

$$\eta^i_{MW}(X) := [X, cone(\eta)(i)[2i]]_{MW} = [\Sigma^\infty X_+, H\widetilde{\mathbb{Z}}/\eta(i)[2i]]_{\mathcal{SH}(k)}.$$

Theorem (Y)

If $R = \mathbb{Z}$ and ${}_2CH^{i+1}(X) = 0$, we have a natural isomorphism

$$\theta^i : CH^i(X) \oplus CH^{i+1}(X) \to \eta^i_{MW}(X).$$

Corollary

If $R = \mathbb{Z}[\frac{1}{2}]$, we have a natural isomorphism

$$\theta^i : CH^i(X)[\frac{1}{2}] \oplus CH^{i+1}(X)[\frac{1}{2}] \to \eta^i_{MW}(X)$$

for any $X \in Sm/k$.

a^k, b^k

Definition

Suppose $n \geq k+1$ and k is odd. Define $a^k, b^k \in \mathbb{Z}$ by

$$\begin{array}{rcl} CH^k(\mathbb{P}^n) \oplus CH^{k+1}(\mathbb{P}^n) & \xrightarrow{\eta^k} & [\mathbb{P}^n, cone(\eta)(k)[2k]]_{MW} \\ (a^k c_1(O(1))^k, b^k c_1(O(1))^{k+1}) & \mapsto & c_n^k \end{array}.$$

They are independent of n.

$c(E)^k : \mathbb{P}(E) \to cone(\eta)(k)[2k]$

Definition

Suppose E is a vector bundle of rank n over X, $R = \mathbb{Z}$, ${}_2CH^*(X) = 0$ and $k - n - 2$ is odd. Define $c(E)^k$ by

$$\begin{array}{ccc} CH^k(\mathbb{P}(E)) \oplus CH^{k+1}(\mathbb{P}(E)) & \xrightarrow{k} & [\mathbb{P}(E), cone(\eta)(k)[2k]]_{MW} \\ (a^k c_1(O(1))^k, b^k c_1(O(1))^{k+1}) & \mapsto & c(E)^k \end{array}.$$

If $R = \mathbb{Z}[\frac{1}{2}]$, $c(E)^k$ is defined for all $X \in Sm/k$.

Projective Orientability

Recall SL^c-bundles are vector bundles E over X such that

$$det(E) \in 2Pic(X).$$

Definition

Let E be an SL^c-bundle with even rank n over X. It's said to be projective orientable if there is an element $th(E) \in \widetilde{CH}^{n-1}(\mathbb{P}(E))$ such that for any $x \in X$, there is a neighbourhood U of x such that $E|_U$ is trivial and

$$th(E)|_U = p^* th_n,$$

where $p : \mathbb{P}^{n-1} \times U \to \mathbb{P}^{n-1}$.

Projective Orientability

- In Chow rings, we can always let $th(E) = c_1(O_{P(E)}(1))^{n-1}$. But this doesn't work for Chow Witt rings!
- If E has a quotient line bundle, it's projective orientable.
- If E has a quotient bundle being projective orientable, it's projective orientable.
- Further characterization?

Projective Bundle Theorem

Theorem (Y)

Let E be a vector bundle of rank n over X. Suppose ${}_2CH^*(X) = 0$ and X admits an open covering $\{U_i\}$ such that $CH^j(U_i) = 0$ for all $j > 0$ and i. Denote by $p : \mathrm{P}(E) \longrightarrow X$.

1. *If n is even and E is projective orientable, the morphism $(p^*, p^* c(E)^{2i-1}, p^* th(E))$*

$$\mathrm{P}(E) \longrightarrow X \oplus \bigoplus_{i=1}^{\frac{n}{2}-1} X \wedge cone(\eta)(2i-1)[4i-2] \oplus X(n-1)[2n-2]$$

   is an isomorphism.

2. *If n is odd, there is an isomorphism*

$$\mathrm{P}(E) \xrightarrow{(p^*, p^* c(E)^{2i-1})} X \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} X \wedge cone(\eta)(2i-1)[4i-2].$$

Projective Bundle Theorem

Corollary

Let E is a vector bundle of odd rank n over X. If X is quasi-projective, we have

$$\mathrm{P}(E) = X \quad \bigoplus_{i=1}^{\frac{n \quad 1}{2}} X \quad cone(\eta)(2i \quad 1)[4i \quad 2].$$

In particular, we have ($k = \min\{\lfloor\frac{i+1}{2}\rfloor, \frac{n-1}{2}\}$)

$$\widetilde{CH}^{i}(\mathrm{P}(E)) = \widetilde{CH}^{i}(X) \quad \bigoplus_{j=1}^{k} \widetilde{CH}^{i-2j+2}(X \quad \mathrm{P}^2)/\widetilde{CH}^{i-2j+2}(X).$$

Projective Bundle Theorem

Theorem (Y)

Let E be a vector bundle of rank n over X. Suppose $2\ 2\ R^{\times}$. Denote by $p : \mathrm{P}(E) \quad !\ X$. If n is even and E is projective orientable, the morphism $(p, p \quad c(E)^{2i-1}, p \quad th(E))$

$$\mathrm{P}(E) \quad !\ X \quad \bigoplus_{i=1}^{\frac{n}{2}-1} X \quad cone(\eta)(2i \quad 1)[4i \quad 2] \quad X(n \quad 1)[2n \quad 2]$$

is an isomorphism.
In particular, we have ($k = min f b\frac{i+1}{2}c, \frac{n}{2} \quad 1g$)

$$\widetilde{CH}^{i}(\mathrm{P}(E)) = \widetilde{CH}^{i}(X) \ \bigoplus_{j=1}^{k} \widetilde{CH}^{i-2j+2}(X \ \ \mathrm{P}^2)/\widetilde{CH}^{i-2j+2}(X) \ \ \widetilde{CH}^{i-n+1}(X)$$

after inverting 2.

Blow-ups

Theorem (Y)

Suppose Z is smooth and closed in X, $n := codim_X(Z)$ is odd and Z is quasi-projective. We have

$$Bl_Z(X) = X \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} Z \otimes cone(\eta)(2i-1)[4i-2].$$

In particular, we have ($k = \min\{\lfloor\frac{i+1}{2}\rfloor, \frac{n-1}{2}\}$)

$$\widetilde{CH}^i(Bl_Z(X)) = \widetilde{CH}^i(X) \oplus \bigoplus_{j=1}^{k} \widetilde{CH}^{i-2j+2}(Z \times \mathrm{P}^2)/\widetilde{CH}^{i-2j+2}(Z).$$

Thank you!