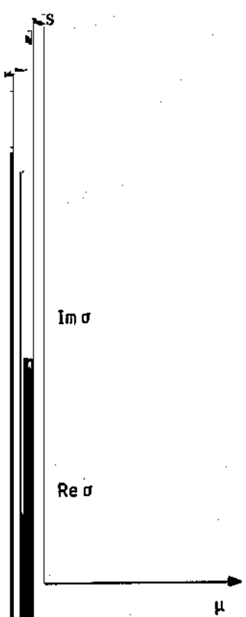

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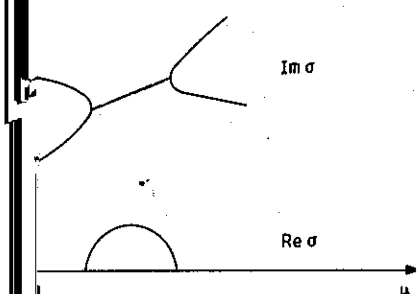

Figure 3.

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for some $n_j \in \mathbb{Z}_+$, and vectors v_{jn} with $v_{j0} = v_j$. Then

$$D^2H(v_1(t), v_2(t)) = \exp[(\sigma_1+\sigma_2)t] \sum_{d=0}^{n_1+n_2} t^d \sum_{n+m=d} D^2H(v_{1n}, v_{2m}) .$$

But, by lemma 1, it is constant. Therefore, if $\sigma_1+\sigma_2 \neq 0$, we get in particular that $D^2H(v_{10}, v_{20}) = 0$, as required.

<u>Definition</u>. If σ is an eigenvalue of z_0, let I_σ be the real invariant space corresponding to eigenvalues $\sigma, -\sigma, \sigma^*, -\sigma^*$.

<u>Lemma 3</u>. If $w = \sum_j v_j$, $v_j \in I_{\sigma_j}$, $(I_{\sigma_j})_j$ distinct, then

$$D^2H(w, w) = \sum_j D^2H(v_j, v_j).$$

<u>Proof</u>. Write $w = \sum_k u_k$, u_k in different eigenspaces E_σ. Then

$$D^2H(w, w) = \sum_{kl} D^2H(u_k, u_l) .$$

By lemma 2, the only terms that contribute are the ones for which $\sigma_k+\sigma_l=0$. So summing up we get $\sum_j D^2H(v_j, v_j)$, as required.

A particular case of this lemma is that <u>the energy of a superposition of waves is the sum over modes of the energy in each mode.</u>

So we can consider the $D^2H|I_\sigma$ separately.

<u>Lemma 4</u>. If σ is non-zero then $D^2H|I_\sigma$ is non-degenerate.

<u>Proof</u>. If $D^2H|I_\sigma$ is degenerate then $\exists v \in I_\sigma$ such that $\forall w \in I_\sigma$, $D^2H(v,w)=0$. But by lemma 2, it is also zero for all $w \in I_{\sigma'}$, for any $I_{\sigma'} \neq I_\sigma$. Therefore it is zero for all $w \in TM_{z_0}$. Thus $J \cdot D^2H\, v = 0$, and v has eigenvalue 0.

Consequently, $D^2H|I_\sigma$ can be expressed as a sum of 2s positive and/or negative squares in an appropriate basis, where 2s is the dimension of I_σ (Lagrange's method, e.g. Gantmacher (1959)). The numbers of positive and negative squares are called the <u>signature</u> of $D^2H|I_\sigma$. If σ has a non-zero real part then we know that $D^2H|I_\sigma$ cannot be definite, because energy conservation would imply that all tangent orbits in I_σ are bounded, contradicting the existence of tangent orbits growing exponentially like $\exp[\pm \mathrm{Re}(\sigma)t]$. In fact:

Let

$$\xi_2 = v_+ - v_- + v_+^* - v_-^* .$$

In either case

$$D^2H(\xi_2,\xi_2) = -D^2H(\xi_1,\xi_1) \quad \text{and} \quad D^2H(\xi_1,\xi_2) = 0.$$

Proceed as before.

Theorem. If all the eigenvalues of an equilibrium z_0 of a Hamiltonian system with Hamiltonian H are pure imaginary and non-zero, and $D^2H_{z_0}|I_\sigma$ is definite for each eigenvalue σ, then z_0 is linearly stable. The equilibrium can lose spectral stability as parameters vary only by collision of eigenvalues for which $D^2H|I_\sigma$ has opposite signature or by collision of eigenvalues at 0.

Proof. The tangent flow decomposes into the direct sum of the flows on the I_σ. Since H is conserved, definiteness of $D^2H|I_\sigma$ implies that all tangent orbits are bounded, proving the first result. As already remarked, a necessary condition to lose spectral stability is the existence of a multiple eigenvalue σ. But if $D^2H|I_\sigma$ is definite then the eigenvalue cannot split into a pair not purely imaginary because for such a pair, $D^2H|I_\sigma$ has equal numbers of positive and negative squares. The signatures of the $D^2H|I_\sigma$ concatenate on collision of eigenvalues, except possibly at zero, because they are non-degenerate. Hence the second result.

This theorem was essentially known by Weierstrass (1858), and appears in a different guise in Wintner (1935) and the appendix to Moser (1968). An analogous result, 'Krein's theorem' holds for stability of periodic orbits of Hamiltonian systems (see Appendix 29 to Arnold and Avez (1968), references therein and Yakubovitch and Starzhinskii (1975)). It can also be extended to the case of weak dissipation: if one adds negative definite dissipation, then the pure imaginary eigenvalues of positive signature move into the left half-plane (damped) while those of negative signature move into the right half-plane (unstable).

4. APPLICATIONS

The importance of the sign of the energy for small disturbances is already well recognised in some areas of physics. We give some examples.

ral example (Weierstrass 1858), if the Hamiltonian splits
sitive definite 'kinetic' part depending quadratically on
t of 'potential' part, depending on q only, with the
c form, then the signatures of all non-zero eigenvalues
xis are positive. This is because for $H(p,q) = K(p) + V(q)$,
responding to a non-zero eigenvalue can have $\delta p = 0$.
$K\ \delta p = 0$, so δq is constant. Similarly for generalised
ce, in the diagonalisation procedure for pure imaginary
generate pairs $K_i(\delta p_i^2 + \delta q_i^2)$. Since the kinetic energy
te, the K_i must all be positive. So there is linear
minima of the potential and it can be lost only by an
to zero. Arnold (1978) shows that the case of a complex
nvalues cannot occur in this case (§23), and analyses
all cases of repeated eigenvalues (Appendix 10).

FINITION OF SIGNATURE

a different definition of signature. We show here that
equivalent to ours. Given a pair σ, $-\sigma$, pure imaginary
se σ to have a positive imaginary part. Then consider

$\in E_\sigma$.

isymmetry of ω. Moser defines the signature of σ and
ure of this quadratic form

), $w \in E_\sigma$ has the same signature as $D^2H|I_\sigma$.

$w \in E_\sigma$ such that $v = w + w^*$, and vice versa. Then since

for $w \in E_\sigma$,

$)^2H(w, w^*)$

$- \sigma^*)\ \omega(w, w^*)$.

is an easier way to calculate the signatures.

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To second order one finds

$$B^2/4 - D = \tfrac{1}{4}(h_{p_1p_1}+h_{p_2p_2}+h_{q_1q_1}+h_{q_2q_2})^2 - \tfrac{1}{4}(h_{p_1p_1}+h_{p_2p_2}-h_{q_1q_1}-h_{q_2q_2})^2$$
$$-(h_{p_1q_1}+h_{p_2q_2})^2 .$$

Thus using Morse's lemma again a crossing in the 'uncoupled' case (last two terms equal to zero) typically develops into a bubble of instability when coupling is added. A more natural normal form, perhaps, for this case is

$$H = \tfrac{1}{2}(q_1^2+p_1^2) - \tfrac{1}{2}(q_2^2+p_2^2)$$

for which, to second order

$$B^2/4-D = \tfrac{1}{4}(h_{p_1p_1}+h_{q_1q_1}+h_{p_2p_2}+h_{q_2q_2})^2 - (h_{p_1p_2}-h_{q_1q_2})^2 - (h_{p_1q_2}+h_{p_2q_1})^2$$

giving the same result. When $B^2/4 - D = 0$, one can check that the Jordan normal form is non-trivial unless all three squares are zero, so we expect the case of a double eigenvalue with mixed signature and diagonal Jordan normal form to be codimension 3. Again this was proved by Galin (1975). As for the definite case, the codimension drops to 2 when one restricts to reversible systems (Jimenez and MacKay 1986). Also one can check that the signatures of the eigenvalues are exchanged as one passes from one of the stable regions to the other.

Lastly, one might wish to know what happens when a pair of imaginary eigenvalues $\pm\sigma$ collide at zero. A normal form for $D^2H|I_\sigma$ with a double eigenvalue $\sigma = 0$ is (Williamson 1936)

$$H(p,q) = a\, p^2/2$$

giving

$$J \cdot D^2H = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$$

which has non-trivial Jordan normal form if $a \neq 0$. The eigenvalues of $J \cdot D^2(H+h)$ are given by

$$\sigma^2 = h_{pq}^2 - a h_{qq} + h_{qq} h_{pp} .$$

So the case of non-trivial Jordan normal form is codimension 1 with

$$\sigma^2 = - a\, h_{qq}$$

to first order. The case of diagonal Jordan normal form gives

$$\sigma^2 = h_{pq}^2 + \tfrac{1}{4}(h_{qq}+h_{pp})^2 - \tfrac{1}{4}(h_{qq}-h_{pp})^2 .$$

As mentioned in the outline, one is not guaranteed persistence of an equilibrium as parameters vary when it has an eigenvalue zero. Thus these last two unfoldings are valid only if the equilibria form smooth submanifolds in the product of a parameter space and the phase space. In fact under the following non-degeneracy conditions on a one-parameter family $H(p,q,\mu)$ with quadratic part $ap^2/2$ at $\mu=0$

$$a \neq 0 \quad H_{qqq} \neq 0 \quad H_{q\mu} \neq 0$$

one gets a smooth path of equilibria

$$\mu \approx - H_{qqq}\, q^2/H_{q\mu}$$

$$p \approx [H_{pqq}H_{q\mu}/2H_{qqq} - H_{p\mu}]\mu/a$$

('tangent bifurcation', 'saddle-node bifurcation', 'limit point'). If $a = 0$ then apart from exceptional cases one can choose coordinates so that in the critical case

$$H(p,q) = p^3 \pm pq^2 .$$

These are codimension 3 and lead to a set of equilibria in their unfoldings called 'umbilics'. We leave it to the standard books on catastrophe theory to describe them.

A complete list of normal forms, codimensions and unfoldings for linear Hamiltonian systems has been given by Galin (1975) (see Arnold 1978, Appendix 6), but the importance of the signature does not seem to have been emphasised there. Neither was the effect of reversibility discussed. Analogous results for linear stability of periodic orbits of Hamiltonian systems will be presented in Howard and MacKay (1986).

7. CONCLUSION

The results described in this chapter should be more widely known and used than they are, in particular now that Hamiltonian formulations are known for so many systems.

CALCULATION OF TRANSPORT COEFFICIENTS IN CHAOTIC SYSTEMS

A S BLAND AND G ROWLANDS

1. INTRODUCTION

It is now recognised that on the one hand complicated behaviour of physical systems can be understood in terms of relatively simple mathematical models whilst on the other hand simple equations can have complicated solutions. A classic example is the logisitc equation

$$x_{n+1} = \lambda x_n(1 - x_n) \ . \tag{1.1}$$

Here λ is a parameter whilst x_n describes the dynamics of the system. This equation describes, with changing λ, a bifurcation sequence eventually leading to chaos - a complicated solution. Feigenbaum has shown the universality of the behaviour and suggested it as a model for the onset of turbulence in physical systems - a complicated behaviour.

An example more germane to the present discussion is that of a charged particle in a spatially non-uniform magnetic field. In dimensionless form we write the equation as

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \qquad \frac{d\mathbf{v}}{dt} = \mathbf{v} \wedge \mathbf{B}(\varepsilon \mathbf{x}) \tag{1.2}$$

where $\mathbf{B}$, the magnetic field, is a given function of space through its dependence on $\mathbf{x}$. We have introduced the parameter ε as a measure of the non-uniformity. For general fields a single constant of the motion exists, namely the kinetic energy v^2. For a uniform magnetic field ($\varepsilon \equiv 0$) the particle undergoes helical motion along the direction of the magnetic field with the perpendicular energy, $v_\perp^2$, and the parallel energy, $v_\parallel^2$, ($v^2 = v_\perp^2 + v_\parallel^2$) independently constant.

For small ε it may be shown that an adiabatic invariant (or magnetic moment), μ, exists such that $\mu(= v_\perp^2/|\mathbf{B}|)$ is a constant to order ε^2. The